

# LONG TIME EXISTENCE OF MINIMIZING MOVEMENT SOLUTIONS OF CALABI FLOW

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ABSTRACT. We recast the Calabi flow in DeGiorgi's language of minimizing movements. We establish the long time existence of minimizing movements for K-energy with arbitrary initial condition. Furthermore we establish some a priori regularity of these solutions, and that sufficiently regular minimizing movements are smooth solutions to Calabi flow.

## 1. INTRODUCTION

Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Fix  $\phi \in C^\infty(M)$  such that  $\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$ , and let  $s_\phi$  denote the scalar curvature of the metric  $\omega_\phi$ . Furthermore, let  $V = \text{Vol}(M)$ , and set  $\bar{s} = \frac{1}{V} \int_M s_\phi \omega_\phi^n$ , which is fixed for any  $\phi$ . A one-parameter family of Kähler potentials  $\phi_t$  is a solution of *Calabi flow* if

$$(1.1) \quad \frac{\partial}{\partial t} \phi = s_\phi - \bar{s}.$$

This flow was introduced by Calabi in his seminal paper [2] on extremal Kähler metrics. Since then several regularity and long time existence results have been obtained. Long time existence and convergence to a metric of constant scalar curvature on Riemann surfaces was shown by Chrusciel [12]. A more direct proof using a concentration/compactness argument was given by Chen [8] (see also [27]). On complex surfaces long time existence and convergence results have been obtained for certain metrics with small energy and toric symmetry [9]. More recent work by Huang [19], [20] approaches the general problem of Calabi flow on toric varieties by exploiting techniques used by Donaldson in understanding the constant scalar curvature equation on such manifolds via a continuity method.

The main motivating conjecture regarding the long time behavior of Calabi flow is simple but ambitious.

**Conjecture 1.1.** (*Calabi, Chen*) *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. The solution to the Calabi flow with any initial condition exists smoothly on  $[0, \infty)$ .*

Furthermore, there are conjectures on the nature of the singularity formation at infinity. One example is the following.

**Conjecture 1.2.** (*Donaldson*) *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold of constant scalar curvature. The solution to the Calabi flow with any initial condition exists for all time and converges to a constant scalar curvature metric.*

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The main purpose of this paper is to prove the long time existence of a certain kind of weak solution to the Calabi flow known as a “minimizing movement” in the terminology of DeGiorgi [14]. These are solutions constructed as limits of time-discretized flows generated by an implicit Euler method. This technique involves freezing the time parameter of the gradient flow and constructing small time-step approximations of the flow as critical points of certain distance-penalizing modifications of the functional in question. This is a very general framework for constructing gradient flows of functionals in metric spaces which has been significantly expanded recently in [1]. We will exploit an earlier instance of this methodology, namely a general existence result of Mayer [24], extending the Crandall-Liggett generation theorem [13] to the setting of metric spaces with nonpositive curvature.

Recall that Calabi flow, while conceived as the gradient flow of the Calabi energy, is also the gradient flow of Mabuchi’s  $K$ -energy functional. As it turns out, the  $K$ -energy on the space of Kähler metrics, denoted  $\mathcal{H}$ , has many of the formal properties which are usually needed in this general setup. For instance, the  $K$ -energy is convex along smooth geodesics, and the space of Kähler metrics, endowed with the Mabuchi metric, has nonpositive curvature, making the distance function also convex. This makes it quite natural to approach existence questions for the Calabi flow using minimizing movements. However, due to the lack of regularity of geodesics and the incompleteness of  $\mathcal{H}$ , care is required in setting things up properly. In preparing our application of Mayer’s theorem a crucial role is played throughout by the theory of geodesics in  $\mathcal{H}$  developed in the work of Chen, [6], [7], Calabi-Chen [4], and Chen-Tian [11].

**Theorem 1.3.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Given  $\phi_0 \in \overline{\mathcal{H}}$  there exists a  $K$ -energy minimizing movement  $\phi : [0, \infty) \rightarrow \overline{\mathcal{H}}$  with initial condition  $\phi_0$ .*

The definition of a  $K$ -energy minimizing movement appears in §4. These solutions come with a host of properties exhibiting the manner in which they can be thought of as gradient lines for  $K$ -energy, and these are shown in §5.3. Note in particular that Theorem 1.3 allows for the definition of a “flow map”  $F : [0, \infty) \times \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$ . This flow map satisfies the semigroup property, and is Hölder continuous of exponent  $\frac{1}{2}$  in the time variable (Theorem 5.22). Furthermore, we show that for all  $t$ ,  $F(t, \cdot) : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$  is distance nonincreasing. This property was shown for smooth solutions to Calabi flow by Calabi-Chen ([4] Theorem 1.3, Theorem 1.5).

**Theorem 1.4.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Given  $\phi_0, \psi_0 \in \overline{\mathcal{H}}$ , for all  $t \geq 0$  one has*

$$d(\phi_t, \psi_t) \leq d(\phi_0, \psi_0).$$

As detailed in §5, this solution guaranteed by Theorem 1.3 is a path through  $\overline{\mathcal{H}}$ , the completion of  $\mathcal{H}$  with respect to the distance topology. This alone guarantees little regularity for  $\phi$  beyond what comes automatically from the regularity of closed positive (1,1) currents. Ideally one would like to show that this minimizing movement solution is in fact smooth, and moreover a solution to Calabi flow. We take two steps in this direction, again exploiting the theory of geodesics in  $\mathcal{H}$ . The first is to establish some

extra regularity for the minimizing movements beyond what is guaranteed by Mayer's theorem.

**Theorem 1.5.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Given  $\phi_0 \in \overline{\mathcal{H}}$ , the  $K$ -energy minimizing movement with initial condition  $\phi_0$  is a map*

$$\phi : [0, \infty) \rightarrow \overline{\mathcal{H}} \cap H_1^2 \cap \text{PSH}(M, J).$$

Lastly we establish the fact that, provided the discretized Calabi flows used to generate the minimizing movement satisfy sufficient a priori estimates, the limiting path through  $\overline{\mathcal{H}}$  is in fact a smooth solution to Calabi flow. See Theorem 7.1 for the precise statement.

**Theorem 1.6.** *A sequence of discrete Calabi flows with step size approaching zero and satisfying sufficient a priori estimates contains a subsequence converging to a smooth solution of Calabi flow.*

**Remark 1.7.** A different approximation scheme for Calabi flow on projective varieties was considered by Fine [16]. Fine's construction defines an ODE on maps defining projective embeddings of the underlying complex manifold, called balancing flow. This is a generalization to the Calabi flow of the techniques used by Donaldson in constructing cscK metrics. The main result shows that this sequence of ODE's, with appropriate initial conditions, converges to a solution to Calabi flow, as long as that solution exists smoothly. To roughly compare this approach to ours, Fine uses a natural sequence of finite dimensional approximations of the space of Kähler metrics to approximate the flow by ODEs, whereas we deal directly with this infinite dimensional space, but discretize the time variable of the flow.

Here is an outline of the rest of the paper. In §2 we review some fundamental facts on the Mabuchi-Semmes-Donaldson metric, and we continue in §3 with a thorough discussion of the structure of geodesics in this metric, including Chen's  $\epsilon$ -geodesics and Chen-Tian's almost smooth geodesics. Then in §4 we define the Moreau-Yosida approximations of  $K$ -energy and set up the notion of a discrete Calabi flow and a minimizing movement for  $K$ -energy. Then in §5 we recall Mayer's theorem and prove Theorem 1.3. Section 6 has the proof of Theorem 1.5, and Theorem 1.6 is proved in §7.

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## 2. THE SPACE OF KÄHLER METRICS

In this section we recall some fundamental properties of the Mabuchi-Semmes-Donaldson metric ([23], [26], [15]) on a Kähler class, and some important functionals on this space. First we recall the definition of the metric and the interpretation of a Kähler class as an infinite dimensional symmetric space with nonpositive curvature.

**Definition 2.1.** Let  $(M^{2n}, \omega, J)$  be a Kähler manifold. Let

$$\mathcal{H} = \{\phi \in C^\infty(M) \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}.$$

This space is often denoted  $\mathcal{H}_{[\omega]}$  to indicate the dependence on the underlying Kähler class, but we have omitted this for notational simplicity, and consider a given compact Kähler manifold with given Kähler class as fixed throughout.

Given  $\phi \in \mathcal{H}$ , one has  $T\mathcal{H}_\phi \cong C^\infty(M)$ . One can use the  $L^2$  inner product associated to  $\phi$  to define an inner product on  $T\mathcal{H}_\phi$ . In particular, given  $f_1, f_2 \in T\mathcal{H}_\phi$ , let

$$\langle f_1, f_2 \rangle_\phi := \int_M f_1 f_2 \omega_\phi^n.$$

This defines a Riemannian metric on  $\mathcal{H}$ . By calculating formally (see §3) one obtains the geodesic equation of a path  $\phi_t \in \mathcal{H}$ :

$$(2.1) \quad \ddot{\phi} = \frac{1}{2} \left| \nabla \dot{\phi} \right|_{\phi(t)}^2,$$

where  $\dot{\phi} = \frac{\partial \phi}{\partial t}$ , etc.

The Levi-Civita connection of this metric is most easily defined in terms of differentiation of vector fields along paths. In particular, if  $\phi(t)$  is a one-parameter family of  $\mathcal{H}$  and  $\psi(t)$  is a smooth family of tangent vectors along this path, we set

$$(2.2) \quad \frac{D}{\partial t} \psi = \dot{\psi} - \frac{1}{2} \left\langle \nabla \psi, \nabla \dot{\phi} \right\rangle_\phi.$$

This connection satisfies the compatibility condition

$$\frac{\partial}{\partial t} \langle \psi_1, \psi_2 \rangle_\phi = \left\langle \frac{D}{\partial t} \psi_1, \psi_2 \right\rangle_\phi + \left\langle \psi_1, \frac{D}{\partial t} \psi_2 \right\rangle_\phi.$$

Amazingly, this metric gives  $\mathcal{H}$  the structure of an infinite dimensional symmetric space of nonpositive curvature. This fact, through its manifestation in Lemma 3.11, plays an essential role in the proof of Theorem 1.3. We record the specific curvature calculation for completeness.

**Lemma 2.2.** ([23] *Theorem 4.3*) *Suppose  $\alpha, \beta \in T_\phi \mathcal{H}$ . Then*

$$R(\alpha, \beta)\gamma = - \{ \{ \alpha, \beta \}_\phi, \gamma \}_\phi$$

where  $\{ \cdot, \cdot \}_\phi$  denotes the Poisson bracket associated to the symplectic manifold  $(M, \omega_\phi)$ . In particular, the sectional curvatures satisfy

$$\langle R(\alpha, \beta)\beta, \alpha \rangle = - \| \{ \alpha, \beta \}_\phi \|_{L^2}^2.$$

**2.1. Decomposition of  $\mathcal{H}$ .** There is a natural decomposition

$$T\mathcal{H}_\phi = \left\{ \psi \left| \int_M \psi dV_\phi = 0 \right. \right\} \oplus \mathbb{R}.$$

We can naturally decompose the entire space  $\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$  according to this decomposition. In particular, let  $\alpha$  be the 1-form on  $\mathcal{H}$  defined by

$$\alpha_\phi(\psi) = \int_M \psi dV_\phi.$$

One can check that  $\alpha$  is closed, and moreover that there exists a unique function  $I : \mathcal{H} \rightarrow \mathbb{R}$  such that  $I(0) = 0$  and  $\alpha = dI$ . By directly integrating one computes

$$(2.3) \quad I(\phi) = \sum_{j=0}^n \frac{1}{(j+1)!(n-j)!} \int_M \phi \omega^{n-j} \wedge (\sqrt{-1} \partial \bar{\partial} \phi)^j.$$

This functional  $I$  has an interpretation as an integral along paths akin to the  $K$ -energy. Specifically, it follows from a calculation in ([5] pg. 615) that in fact

$$(2.4) \quad I(\phi) = \frac{1}{V} \int_0^1 \int_M \dot{\phi} \omega_\phi^n dt$$

where  $\phi : [0, 1] \rightarrow \mathcal{H}$  is a smooth one-parameter family satisfying  $\phi_0 = 0$ ,  $\phi_1 = \phi$ .

**Remark 2.3.** It follows from (2.4) that the Calabi flow preserves  $\mathcal{H}_0$ , and thus it is natural to restrict our attention entirely to this space.

**2.2. Functionals on  $\mathcal{H}$ .** In this subsection we recall some functionals on  $\mathcal{H}$  and some of their properties for convenience.

**Definition 2.4.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. The  $K$ -energy functional is

$$(2.5) \quad \nu(\phi) = - \int_0^1 \int_M (s(\omega_\phi) - \bar{s}) \dot{\phi} \omega_\phi^n dt,$$

where  $\phi : [0, 1] \rightarrow \mathcal{H}$  is a smooth one-parameter family satisfying  $\phi_0 = 0$ ,  $\phi_1 = \phi$ .

**Remark 2.5.** Herein we do not concern ourselves with the “modified”  $K$ -energy which is used in the presence of a continuous group of automorphisms of  $(M, J)$  (see [3], [17]). The results of this paper apply to constructing minimizing movements for this modified  $K$ -energy as well, although we do not explicitly do this here. We have a very brief discussion of the convergence properties of minimizing movements below, and it is likely that use of the modified  $K$ -energy will be necessary in deriving more precise convergence results.

**Definition 2.6.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Let

$$(2.6) \quad J(\phi) = - \frac{1}{(n-1)!} \int_0^1 \int_M \dot{\phi} \rho(\omega) \wedge \omega_{\phi_t}^{n-1},$$

where  $\phi : [0, 1] \rightarrow \mathcal{H}$  is a smooth one-parameter family satisfying  $\phi_0 = 0$ ,  $\phi_1 = \phi$ . By integrating along a straight line path one obtains the explicit formula

$$(2.7) \quad J(\phi) = - \sum_{j=0}^{n-1} \frac{1}{(j+1)!(n-j-1)!} \int_M \phi \rho(\omega) \wedge \omega^{n-j-1} \wedge (\sqrt{-1} \partial \bar{\partial} \phi)^j.$$

**Definition 2.7.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Let

$$(2.8) \quad I^A(\phi) = \frac{1}{V} \int_M \phi (\omega^n - \omega_\phi^n) = \frac{1}{V} \sum_{i=0}^{n-1} \int_M \sqrt{-1} \partial \phi \wedge \bar{\partial} \phi \wedge \omega^i \wedge \omega_\phi^{n-1-i},$$

$$(2.9) \quad J^A(\phi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1} \partial \phi \wedge \bar{\partial} \phi \wedge \omega^i \wedge \omega_\phi^{n-1-i}.$$

**Remark 2.8.** We have decorated these functionals with the superscript  $A$ , since these functionals typically are denoted by  $I$  and  $J$ , but so unfortunately are the functionals defined in (2.4), (2.6).

These functionals satisfy the inequality

$$\frac{1}{n} J^A(\phi) \leq I^A(\phi) - J^A(\phi) \leq n J^A(\phi).$$

The next lemma gives an explicit form for the  $K$ -energy, which is arrived at by evaluating the definition (2.5) along linear paths. This formulation has the further benefit of showing that the  $K$ -energy is well-defined for  $C^{1,1}$  limits of metrics.

**Lemma 2.9.** ([5] pg. 1) *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Then*

$$(2.10) \quad \nu(\phi) = \int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n + J(\phi) + \bar{s} I(\phi).$$

Next we record a crucial observation regarding the  $K$ -energy, namely its convexity along smooth geodesics in  $\mathcal{H}$ .

**Lemma 2.10.** ([22] Theorem 6.2) *Let  $\phi_t$  be a path in  $\mathcal{H}$ . Then*

$$\frac{d^2 \nu(\phi)}{dt^2} = \frac{1}{V} \left\| \bar{\partial} \nabla^{1,0} \dot{\phi} \right\|_{L^2(\omega_\phi)}^2 - \int_M \left( \ddot{\phi} - \frac{1}{2} |\nabla \dot{\phi}|^2 \right) (s_\phi - \bar{s}) dV_\phi.$$

*In particular, the  $K$ -energy is weakly convex along a smooth geodesic.*

### 3. THE STRUCTURE OF GEODESICS

**3.1. Approximate geodesics and properties of the distance function.** In this section we recall various results on geodesics in the space of Kähler metrics. We begin with the basic definitions and some variational formulas. Then we recall Chen's theory of  $\epsilon$ -approximate geodesics [6], which suffice for proving most statements concerning the distance function on  $\mathcal{H}$ . In the next subsection we will recall some fundamental facts about the more profound regularity results concerning the “partially smooth” and “almost smooth” geodesics of [11]. The starting point is to define the infinitesimal energy element for a path in  $\mathcal{H}$ . The fundamental definitions are formally identical to those from Riemannian geometry.

**Definition 3.1.** Let  $\gamma : [0, 1] \rightarrow \mathcal{H}$  be a smooth path. The *energy element along  $\gamma$*  is

$$E(t) := \int_M \left| \frac{\partial \gamma}{\partial t}(t) \right|^2 \omega_{\gamma(t)}^n.$$

**Definition 3.2.** Given  $\gamma : [0, 1] \rightarrow \mathcal{H}$  a smooth path, the *length* of  $\gamma$  is

$$\mathcal{L}(\gamma) := \int_0^1 E(t)^{\frac{1}{2}} dt.$$

Moreover, the *energy* of a path in  $\mathcal{H}$  is

$$\mathcal{E}(\gamma) := \int_0^1 E(t) dt.$$

**Definition 3.3.** Given  $\phi, \psi \in \mathcal{H}$ , the *distance* from  $\phi$  to  $\psi$  is

$$d(\phi, \psi) = \inf_{\gamma: \phi \rightarrow \psi} \mathcal{L}(\gamma).$$

We next compute the first variation of the length integral. This was done in ([22] Theorem 7.3) for endpoint-fixed variations, which will not suffice for our purposes. While the calculation is formally identical to the usual calculation in Riemannian geometry, we include it for convenience.

**Lemma 3.4.** Let  $\gamma_i : [0, 1] \rightarrow \mathcal{H}, i = 1, 2$  be smooth paths, and suppose

$$\gamma(t, s) : [0, 1] \times [0, 1] \rightarrow \mathcal{H}$$

is a smooth family of curves connecting  $\gamma_1(s)$  to  $\gamma_2(s)$ . Then

$$\begin{aligned} \frac{d\mathcal{L}(\gamma(\cdot, s))}{ds} &= E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle_{\gamma(t, s)} \Big|_{t=0}^{t=1} \\ &\quad + \int_0^1 E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial t}, \frac{\langle \langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \rangle \rangle}{\langle \langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \rangle \rangle} \frac{\partial \gamma}{\partial t} - \frac{\partial \gamma}{\partial s} \right\rangle \right\rangle dt. \end{aligned}$$

*Proof.* We directly compute

$$\begin{aligned} \frac{d\mathcal{L}(\gamma(\cdot, s))}{ds} &= \frac{d}{ds} \int_0^1 \left[ \left\langle \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle^{\frac{1}{2}} \right] dt \\ &= \int_0^1 E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{D}{\partial s} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle dt \\ &= \int_0^1 E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle dt \\ &= \int_0^1 E(s, t)^{-\frac{1}{2}} \left[ \frac{d}{dt} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle - \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{D}{\partial t} \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \right] dt \\ &= \int_0^1 \left[ \frac{d}{dt} \left( E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle + E(s, t)^{-\frac{3}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \left\langle \left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \right. \right. \\ &\quad \left. \left. - E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{D}{\partial t} \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \right] dt \\ &= E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \Big|_{t=0}^{t=1} + \int_0^1 E(s, t)^{-\frac{1}{2}} \left\langle \left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial t}, \frac{\langle \langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \rangle \rangle}{\langle \langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \rangle \rangle} \frac{\partial \gamma}{\partial t} - \frac{\partial \gamma}{\partial s} \right\rangle \right\rangle dt. \end{aligned}$$

□

This lemma in particular establishes the geodesic equation (2.1). However, as mentioned above, the lack of full  $C^\infty$  regularity results for geodesic paths in  $\mathcal{H}$  is an essential difficulty in working with  $\mathcal{H}$ . Nonetheless, by exploiting  $\epsilon$ -approximate geodesics, which we will define below, one can establish foundational results on the metric space structure of  $\mathcal{H}$ .

**Definition 3.5.** Let  $(M^{2n}, \omega, J)$  a compact Kähler manifold, and fix  $\Omega$  a background volume form on  $M$ . A one-parameter family  $\gamma : [0, 1] \rightarrow \mathcal{H}$  is an  $\epsilon$ -approximate geodesic if

$$(3.1) \quad \left( \ddot{\gamma} - |\nabla \dot{\gamma}|_\gamma^2 \right) \omega_\gamma^n = \epsilon \Omega.$$

**Lemma 3.6.** ([6] Lemma 7) Suppose  $\gamma_i : [0, 1] \rightarrow \mathcal{H}, i = 1, 2$  are two smooth curves. There exists  $\epsilon_0$  small and a smooth two-parameter family of curves

$$\gamma(t, s, \epsilon) : [0, 1] \times [0, 1] \times (0, \epsilon_0] \rightarrow \mathcal{H}$$

such that

- (1) For fixed  $s_0$ , the curve  $\gamma(t, s_0, \epsilon)$  is an  $\epsilon$ -approximate geodesic from  $\gamma_1(s)$  to  $\gamma_2(s)$ .
- (2) There exists  $C$  depending on  $\{\gamma_i\}$  such that

$$|\gamma| + \left| \frac{\partial \gamma}{\partial s} \right| + \left| \frac{\partial \gamma}{\partial t} \right| < C, \quad 0 \leq \frac{\partial^2 \gamma}{\partial t^2} < C, \quad \frac{\partial^2 \gamma}{\partial s^2} < C.$$

- (3) For fixed  $s_0$ , the curves  $\{\gamma(t, s_0, \epsilon)\}$  converge as  $\epsilon \rightarrow 0$  to the unique geodesic connecting  $\gamma_1(s)$  to  $\gamma_2(s)$  in the weak  $C^{1,1}$  topology.
- (4) There exists a constant  $C$  depending on  $\{\gamma_i\}$  such that

$$\left| \frac{\partial E}{\partial t} \right| \leq \epsilon C.$$

That is, the energy element converges to a constant along each curve  $\gamma(t, s_0, \epsilon)$  as  $\epsilon \rightarrow 0$ .

This lemma in particular yields the existence of a  $C^{1,1}$  geodesic connecting any two points. A final point required in using these geodesics to prove that  $(\mathcal{H}, d)$  is a metric space is to bound their length from below, and we record this estimate as we will use it several times below.

**Lemma 3.7.** Given  $\psi, \phi \in \mathcal{H}$  with  $I(\psi) = I(\phi) = 0$ , then

$$d(\psi, \phi) \geq V^{-\frac{1}{2}} \max \left\{ \int_{\phi - \psi > 0} (\phi - \psi) \omega_\phi^n, - \int_{\phi - \psi < 0} (\phi - \psi) \omega_\psi^n \right\}.$$

*Proof.* By ([6] Corollary 3) it suffices to estimate the length of the  $C^{1,1}$  geodesic connecting  $\phi$  to  $\psi$ . This proof follows ([6] Proposition 2) but with a more general base



point. Let  $\tilde{\gamma}(t) = t\phi + (1-t)\psi$ , and let  $a(t) = I(\tilde{\gamma}(t))$ . It follows from (2.4) that

$$\begin{aligned}\dot{a}(t) &= \int_M (\phi - \psi) \omega_{\tilde{\gamma}(t)}^n, \\ \ddot{a}(t) &= \int_M (\phi - \psi) \Delta_{\tilde{\gamma}(t)} (\phi - \psi) \omega_{\tilde{\gamma}(t)}^n \leq 0.\end{aligned}$$

Hence  $\dot{a}(0) \geq a(1) - a(0) \geq \dot{a}(1)$ , thus

$$(3.2) \quad \int_M (\phi - \psi) \omega_\psi^n \geq I(\phi) - I(\psi) \geq \int_M (\phi - \psi) \omega_\phi^n$$

Since  $I(\phi) = I(\psi) = 0$  we conclude that  $\phi - \psi$  attains positive and negative values. Now let  $\gamma_t$  be an  $\epsilon$ -approximate geodesic connecting  $\psi$  to  $\phi$ . Note that one has  $\ddot{\gamma}(t) > \frac{1}{2} |\nabla \dot{\gamma}|_\gamma^2 \geq 0$ . It follows that

$$(3.3) \quad \dot{\gamma}(0) \leq \phi - \psi \leq \dot{\gamma}(1).$$

Let  $E(t)$  denote the infinitesimal energy along  $\gamma(t)$ . It follows from Hölder's inequality and (3.3) that

$$\begin{aligned}\sqrt{E(1)} &\geq V^{-\frac{1}{2}} \int_M |\dot{\gamma}(1)| \\ &\geq V^{-\frac{1}{2}} \int_{\dot{\gamma}(1) > 0} \dot{\gamma}(1) \omega_\phi^n \\ &\geq V^{-\frac{1}{2}} \int_{\phi - \psi > 0} (\phi - \psi) \omega_\phi^n.\end{aligned}$$

Similarly one can derive

$$\sqrt{E(0)} \geq -V^{-\frac{1}{2}} \int_{\phi - \psi < 0} (\phi - \psi) \omega_\psi^n.$$

Note that by Lemma 3.6 part (4) it follows that for all  $t_1, t_2 \in [0, 1]$  one has  $|E(t_1) - E(t_2)| \leq C\epsilon$ . Hence for all  $t$ ,

$$\sqrt{E(t)} \geq V^{-\frac{1}{2}} \max \left\{ \int_{\phi - \psi > 0} (\phi - \psi) \omega_\phi^n, - \int_{\phi - \psi < 0} (\phi - \psi) \omega_\psi^n \right\} - C\epsilon.$$

Integrating this inequality from 0 to 1 and sending  $\epsilon$  to zero yields the result.  $\square$

**Theorem 3.8.** ([6] Corollary 3, Theorem 6)

- (1) The space of Kähler potentials  $\mathcal{H}$  is convex by  $C^{1,1}$  geodesics.
- (2)  $(\mathcal{H}, d)$  is a metric space.

Given that the geodesics between points in  $\mathcal{H}$  pass through the closure of  $\mathcal{H}$  with respect to weak  $C^{1,1}$  convergence, it is convenient to work directly with this space. As we will also need to work with the metric space completion of  $\mathcal{H}$ , we take the time here to define notation for these two spaces. Note that in most of the literature  $\overline{\mathcal{H}}$  denotes the  $C^{1,1}$  closure, whereas for us this denotes the metric space completion.

**Definition 3.9.** Given  $(M^{2n}, \omega, J)$  a compact Kähler manifold, let  $\mathcal{H}^{1,1}$  denote the closure of  $\mathcal{H}$  with respect to weak  $C^{1,1}$  convergence. Furthermore, let  $(\overline{\mathcal{H}}, \overline{d})$  denote the metric space completion of  $(\mathcal{H}, d)$ . Note that weak  $C^{1,1}$  convergence implies strong  $C^{1,\alpha}$  convergence, which in turn by a Lemma 3.10 implies convergence in the distance topology. This means that one can interpret a point in  $\phi \in \mathcal{H}^{1,1}$  as a point in  $\overline{\mathcal{H}}$  by mapping it to the equivalence class of Cauchy sequences represented by any sequence converging to  $\phi$  in the weak  $C^{1,1}$  topology. We will do this implicitly in various points in the paper.

**Lemma 3.10.** *Let  $\{\phi_n\} \in \mathcal{H}$  be a sequence converging in the weak  $C^{1,1}$  topology. Then  $\{\phi_n\}$  is a Cauchy sequence in  $(\mathcal{H}, d)$ .*

*Proof.* Convergence in the weak  $C^{1,1}$  topology implies uniform convergence of the potential functions. Fix  $m, n \in \mathbb{N}$  and set  $\gamma_{m,n}(t) = t\phi_n + (1-t)\phi_m$ . We estimate

$$\begin{aligned} \mathcal{E}(\gamma_{n,m}) &= \int_0^1 \int_M (\phi_n - \phi_m)^2 \omega_{t\phi_n + (1-t)\phi_m}^n dt \\ &\leq |\phi_n - \phi_m|_{C^0}^2 \int_0^1 \int_M \omega_{t\phi_n + (1-t)\phi_m}^n dt \\ &= |\phi_n - \phi_m|_{C^0}^2 V. \end{aligned}$$

Due to the uniform convergence in  $C^0$ , the lemma follows.  $\square$

Furthermore, as mentioned above in Lemma 2.2, the space  $\mathcal{H}$  formally has non-positive curvature. By again exploiting the theory of approximate geodesics one can exhibit the nonpositivity of curvature in the sense of Alexandrov, due to Calabi-Chen [4]. We include a discussion of the proof to make precise the sense in which it holds with respect to points in  $\overline{\mathcal{H}}$ .

**Lemma 3.11.** ([4] Theorem 1.1) *Fix  $a, b, c \in \mathcal{H}$ . For any  $s, 0 \leq s \leq 1$ , let  $p_s \in \overline{\mathcal{H}}$  denote the point on the geodesic path connecting  $b$  to  $c$  satisfying  $d(b, p_s) = sd(b, c)$  and  $d(p_s, c) = (1-s)d(b, c)$ . Then*

$$(3.4) \quad \overline{d}(a, p_s)^2 \leq (1-s)d(a, b)^2 + sd(a, c)^2 - s(1-s)d(b, c)^2.$$

*Proof.* Let  $\gamma^\epsilon : [0, 1] \rightarrow \mathcal{H}$  denote the  $\epsilon$ -approximate geodesic connecting  $b$  to  $c$ , and let  $\mathcal{E}(s)$  denote the energy of the  $\epsilon$ -approximate geodesic connecting  $a$  to  $\gamma^\epsilon(s)$ . Since the curvature of  $\mathcal{H}$  is nonpositive, Calabi-Chen use Jacobi field estimates to estimate the second derivative of  $\mathcal{E}(s)$  from below and obtain the inequality ([4] pg. 185)

$$(3.5) \quad \mathcal{E}(s) \leq (1-s)\mathcal{E}(0) + s\mathcal{E}(1) - s(1-s)(\mathcal{E}(\gamma^\epsilon) - C\epsilon).$$

Fix  $s > 0$ . By Lemma 3.6 we know that for any sequence  $\{\epsilon_i\} \rightarrow 0$ ,  $\{\gamma^{\epsilon_i}(s)\}$  is a Cauchy sequence in  $\mathcal{H}$ , and moreover  $\lim_{i \rightarrow \infty} d(b, \gamma^{\epsilon_i}(s)) = sd(b, c)$  and  $\lim_{i \rightarrow \infty} d(\gamma^{\epsilon_i}(s), c) = (1-s)d(b, c)$ . In other words this Cauchy sequence represents the point  $p_s \in \overline{\mathcal{H}}$  on the geodesic connecting  $b$  to  $c$ . Moreover, the energies  $\mathcal{E}(s)$  converge as  $\epsilon \rightarrow 0$  to the squared distance from  $a$  to  $\gamma^\epsilon(s)$ , so again by the definition of the metric on  $\overline{\mathcal{H}}$ , the left hand side of (3.5) converges as  $\epsilon \rightarrow 0$  to  $\overline{d}(a, p_s)$ . The lemma follows.  $\square$

Lastly in this subsection we record Chen's theorem on the decay of  $K$ -energy with distance which is crucial to what follows. We give a proof in the simple case of two points connected by a smooth geodesic, to exhibit why the estimate takes the form it does.

**Theorem 3.12.** ([7] Theorem 1.2) *Let  $\phi_0, \phi_1 \in \mathcal{H}$ . Then*

$$\nu(\phi_1) \geq \nu(\phi_0) - d(\phi_0, \phi_1) \sqrt{\mathcal{C}(\phi_0)}.$$

*Proof.* Fix  $\phi_0, \phi_1 \in \mathcal{H}$ , and suppose  $\phi : [0, 1] \rightarrow \mathcal{H}$  is a smooth geodesic connecting  $\phi_0$  to  $\phi_1$ . We note that

$$\left. \frac{\partial \nu}{\partial t} \right|_{t=0} = - \int_M (s_\phi - \bar{s}) \dot{\phi} \omega_\phi^n \geq -\sqrt{\mathcal{C}(\phi_0)} \left\| \dot{\phi} \right\|_{\omega_\phi} = -\sqrt{\mathcal{C}(\phi_0)} d(\phi_0, \phi_1)$$

as the geodesic  $\phi_t$  has constant speed. But, by Lemma 2.10 we conclude that for all  $t \in [0, 1]$ ,

$$\frac{\partial \nu}{\partial t}(t) \geq -\sqrt{\mathcal{C}(\phi_0)} d(\phi_0, \phi_1).$$

Integrating over  $[0, 1]$  it follows that

$$\nu(\phi_1) \geq \nu(\phi_0) - d(\phi_0, \phi_1) \sqrt{\mathcal{C}(\phi_0)}.$$

The rigorous proof requires the theory of almost smooth geodesics of Chen-Tian [11] discussed in §3.3.  $\square$

**3.2. Variational properties.** We will require the variation of geodesic distance as well as a convexity property for the distance function, which we record below.

**Lemma 3.13.** *The distance function is  $C^1$ . Specifically, let  $\phi(s)$  denote a path in  $\mathcal{H}$  and  $L(s)$  denote the geodesic distance from 0 to  $\phi(s)$ , then*

$$\frac{dL}{ds} = E(1, s)^{-\frac{1}{2}} \left\langle \left\langle \frac{d\phi}{ds}, \frac{\partial \gamma(1, s)}{\partial t} \right\rangle \right\rangle = \left[ \int_M \frac{\partial \gamma(1, s)}{\partial t} \frac{d\phi}{ds} dV_s \right] \left[ \int_M \left| \frac{\partial \gamma(1, s)}{\partial t} \right|^2 dV_s \right]^{-\frac{1}{2}}$$

where  $\gamma(t, s) : [0, 1] \rightarrow \mathcal{H}$  is the unique  $C^{1,1}$  geodesic from 0 to  $\phi(s)$ .

*Proof.* This is contained in ([6] Theorem 6), but we include the proof since the derivative itself is not stated separately therein and moreover we will use this calculation later. With  $\phi_1(s) \equiv 0$  for all  $s$  and  $\phi_2(s) = \phi(s)$  the given path, let  $\gamma(t, s, \epsilon)$  denote the family of approximate geodesics guaranteed by Lemma 3.6. Furthermore, set

$$(3.6) \quad \mathcal{L}(s, \epsilon) := \mathcal{L}(\gamma(\cdot, s, \epsilon)).$$

As  $\gamma(t, s, \epsilon)$  is a smooth family, we may apply Lemma 3.4 and the  $\epsilon$ -approximate geodesic equation to yield

$$\begin{aligned} \frac{d\mathcal{L}(s, \epsilon)}{ds} &= E(t, s, \epsilon)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle \Big|_{t=0}^{t=1} \\ &\quad + \int_0^1 E(t, s, \epsilon)^{-\frac{1}{2}} \left\langle \left\langle \frac{D}{\partial t} \frac{\partial \gamma}{\partial t}, \frac{\langle \langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \rangle \rangle}{\langle \langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \rangle \rangle} \frac{\partial \gamma}{\partial t} - \frac{\partial \gamma}{\partial s} \right\rangle \right\rangle dt \\ &= E(1, s, \epsilon)^{-\frac{1}{2}} \left\langle \left\langle \frac{d\phi}{ds}, \frac{\partial \gamma(1, s)}{\partial t} \right\rangle \right\rangle \\ &\quad + \epsilon \int_0^1 E(t, s, \epsilon)^{-\frac{1}{2}} \int_M \left( \frac{\langle \langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \rangle \rangle}{\langle \langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \rangle \rangle} \frac{\partial \gamma}{\partial t} - \frac{\partial \gamma}{\partial s} \right) \Omega dt \end{aligned}$$

Integrating from  $s_1$  to  $s_2$  and dividing by  $s_2 - s_1$  yields

$$\begin{aligned} &\left| \frac{\mathcal{L}(s_2, \epsilon) - \mathcal{L}(s_1, \epsilon)}{s_2 - s_1} - \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} E(s, 1, \epsilon)^{-\frac{1}{2}} \left\langle \left\langle \frac{d\phi}{ds}, \frac{\partial \gamma(1, s)}{\partial t} \right\rangle \right\rangle ds \right| \\ &\leq \frac{\epsilon}{s_2 - s_1} \int_{s_1}^{s_2} \int_0^1 E(t, s, \epsilon)^{-\frac{1}{2}} \int_M \left( \frac{\langle \langle \frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \rangle \rangle}{\langle \langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial t} \rangle \rangle} \frac{\partial \gamma}{\partial t} - \frac{\partial \gamma}{\partial s} \right) \Omega dt ds \\ &\leq C\epsilon. \end{aligned}$$

The final estimate follows from Lemma 3.6. In particular, as the energy element  $E$  approaches a constant it is in particular bounded below, and the integrands and volume forms are uniformly bounded as well. Taking the limit as  $\epsilon$  goes to zero yields

$$\begin{aligned} \lim_{s_2 \rightarrow s_1} \frac{\mathcal{L}(s_2) - \mathcal{L}(s_1)}{s_2 - s_1} &= \lim_{s_2 \rightarrow s_1} \int_{s_1}^{s_2} E(s, 1)^{-\frac{1}{2}} \left\langle \left\langle \frac{\partial \gamma(1, s)}{\partial t}, \frac{d\phi}{ds} \right\rangle \right\rangle ds \\ &= E(1, s)^{-\frac{1}{2}} \left\langle \left\langle \frac{d\phi}{ds}, \frac{\partial \gamma(1, s)}{\partial t} \right\rangle \right\rangle \\ &= \int_M \frac{\partial \gamma(1, s)}{\partial t} \frac{d\phi}{ds} dV_s \left[ \int_M \left| \frac{\partial \gamma(1, s)}{\partial t} \right|^2 dV_s \right]^{-\frac{1}{2}}. \end{aligned}$$

□

**3.3. Higher regularity of geodesics.** In this subsection we recall the improved regularity theory of [11]. We begin by recalling the interpretation of the geodesic equation in terms of the homogeneous complex Mongé-Ampère (HCMA) equation.

**Lemma 3.14.** *Let  $\phi : [0, 1] \rightarrow \mathcal{H}$  be a continuous path. Extend this to a function  $\phi : [0, 1] \times S^1 \rightarrow \mathcal{H}$  via  $\phi(t, \theta, x) = \phi(t)(x)$ . Then  $\phi$  is a geodesic if and only if*

$$(3.7) \quad (\pi_2^* \omega + \sqrt{-1} \partial \bar{\partial} \phi)^{n-1} = 0 \quad \text{on } \Sigma \times M,$$

where  $\Sigma = [0, 1] \times S^1$  and  $\pi_i$  are the natural projection operators to  $\Sigma$  and  $M$ .

Chen's fundamental existence theorem for  $C^{1,1}$  geodesics in  $\mathcal{H}$  is actually more general, and applies to more general solutions to (3.7).

**Theorem 3.15.** ([6] §3) *For a smooth map  $\phi_0 : \partial\Sigma \rightarrow \mathcal{H}$ , there exists a unique  $C^{1,1}$  solution  $\phi$  of (3.7) such that  $\phi = \phi_0$  on  $\partial\Sigma$  and  $\phi(z, \cdot) \in \mathcal{H}^{1,1}$  for each  $z \in \Sigma$ .*

**Definition 3.16.** Suppose  $\phi$  is a  $C^{1,1}$  solution of (3.7). The *regular part* of  $\phi$  is

$$\mathcal{R}_\phi = \left\{ (z, x) \in \Sigma \times M \mid \exists U, (z, x) \in U, \phi|_U \in C^\infty, \text{ and } \omega_\phi|_{\{z\} \times M} > 0 \right\}.$$

Inside  $\mathcal{R}_\phi$  we can define a distribution

$$\mathcal{D}_\phi|_{(z,x)} = \{v \in T_z\Sigma \times T_xM \mid i_v(\pi_2^*\omega + \sqrt{-1}\partial\bar{\partial}\phi) = 0\}.$$

Note that since  $\omega$  is closed  $\mathcal{D}_\phi$  is integrable. Given a subset  $\mathcal{V} \subset \Sigma \times M$ , we say that  $\mathcal{R}_\phi$  is *saturated* in  $\mathcal{V}$  if every maximal integral submanifold of  $\mathcal{D}_\phi$  in  $\mathcal{R}_\phi \cap \mathcal{V}$  is a disk and closed in the subspace topology of  $\mathcal{V}$ . Note that since  $\omega_\phi > 0$  in  $\mathcal{R}_\phi$ ,  $\mathcal{D}_\phi \cap T^{1,0}\Sigma \times T^{1,0}M$  is one dimensional, and there is a projection of a unit length generator of this space onto  $T^{1,0}M$ . We call this vector field  $X$ .

**Definition 3.17.** A solution  $\phi$  of (3.7) is *partially smooth* if it satisfies the following conditions:

- (1) It has a uniform  $C^{1,1}$  bound on  $\Sigma \times M$  and  $\mathcal{R}_\phi$  is saturated in  $\Sigma \times M$ .
- (2)  $\mathcal{R}_\phi \cap (\partial\Sigma \times M)$  is open and dense in  $\partial\Sigma \times M$ .
- (3) The volume form  $\omega_\phi^n$  can be extended to  $\overset{\circ}{\Sigma} \times M$  as a continuous  $(n, n)$  form, where  $\overset{\circ}{\Sigma} = \Sigma \setminus \partial\Sigma$ .

**Theorem 3.18.** ([11] Theorem 1.3.2) *Suppose  $\Sigma$  is a unit disc. Given  $\phi_0 : \partial\Sigma \rightarrow \mathcal{H}$  a smooth map, there exists a unique partially smooth solution to (3.7).*

**Definition 3.19.** A solution  $\phi$  of (3.7) is *almost smooth* if it satisfies the following conditions:

- (1)  $\phi$  is partially smooth
- (2)  $\mathcal{D}_\phi$  extends to a continuous distribution in an open dense saturated set  $\tilde{\mathcal{V}} \subset \Sigma \times M$  such that  $\tilde{\mathcal{S}}_\phi := \Sigma \times M \setminus \tilde{\mathcal{V}}$  has the Whitney extension property
- (3) The leaf vector field  $X$  is uniformly bounded in  $\tilde{\mathcal{V}}$ .

**Remark 3.20.** The set  $\tilde{\mathcal{S}}_\phi$  is the *singular part* of  $\phi$ , and in general  $(\Sigma \times M \setminus \mathcal{R}_\phi) \setminus \tilde{\mathcal{S}}_\phi \neq \emptyset$ .

**Theorem 3.21.** ([11] Theorem 1.3.4) *Suppose  $\Sigma$  is a unit disc. Given  $\phi_0 : \partial\Sigma \rightarrow \mathcal{H}$ ,  $\phi_0 \in C^{k,\alpha}$ ,  $k \geq 2$ ,  $0 < \alpha < 1$ , and given  $\epsilon > 0$ , there exists  $\phi_\epsilon : \partial\Sigma \rightarrow \mathcal{H}$  and an almost smooth solution to (3.7) with boundary value  $\phi_\epsilon$  such that*

$$\|\phi_0 - \phi_\epsilon\|_{C^{k,\alpha}(\partial\Sigma \times M)} < \epsilon.$$

One crucial application of this theory of partially/almost smooth geodesics is to establish the convexity of the  $K$ -energy in a more general setting.

**Theorem 3.22.** ([11] Corollary 6.1.2) *Suppose  $\phi : \Sigma \rightarrow \mathcal{H}^{1,1}$  is a partially smooth solution to (3.7). Then the induced  $K$ -energy function  $\nu : \Sigma \rightarrow \mathbb{R}$  is a bounded weakly sub-harmonic function in  $\Sigma$ .*

This is a deep, technical result which lies at the heart of Chen-Tian's proof of uniqueness of cscK metrics. As will become clear in §5 it is crucial to the proof of Theorem 1.3 as well.

#### 4. DEFINITION OF MINIMIZING MOVEMENTS

In this section we give the setup for constructing minimizing movement solutions of  $K$ -energy. To begin with we precisely define the functional for which we are constructing minimizing movements.

**Definition 4.1.** Let  $(X, d)$  be a metric space and  $f : X \rightarrow \mathbb{R}$  a lower semicontinuous function. If  $(\overline{X}, \overline{d})$  denotes the completion of  $(X, d)$ , we define the *lower semicontinuous extension of  $f$*  by

$$(4.1) \quad \overline{f}(x) := \begin{cases} f(x) & x \in X \\ \liminf_{x_n \rightarrow x, \{x_n\} \in X} f(x_n) & x \in \overline{X} \setminus X. \end{cases}$$

**Definition 4.2.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Recall from Definition 3.9 that  $\overline{\mathcal{H}}$  denotes the metric space completion of  $(\mathcal{H}, d)$ . We set

$$\overline{\nu} : \overline{\mathcal{H}} \rightarrow \mathbb{R}$$

to be the lower semicontinuous extension of  $\nu : \mathcal{H} \rightarrow \mathbb{R}$  in the sense of Definition 4.1. A simple lemma (see Lemma 5.10 below) shows that  $\overline{\nu}$  is indeed lower semicontinuous.

**Remark 4.3.** A subtle point in this definition is that, while  $\nu$  is well-defined for  $\phi \in \mathcal{H}^{1,1}$  by Lemma 2.10, it does not necessarily hold that  $\nu(\phi) = \overline{\nu}(\phi)$  for all  $\phi \in \mathcal{H}^{1,1}$ . The reason for defining  $\overline{\nu}$  as the extension of  $\nu$  as defined on  $\mathcal{H}$ , instead of  $\mathcal{H}^{1,1}$ , is that it is not clear that  $\nu$  is lower semicontinuous on  $\mathcal{H}^{1,1}$ . Indeed, lower semicontinuity for  $\nu$  on  $\mathcal{H}$  follows from Theorem 3.12, where the Calabi energy of a point in  $\mathcal{H}$  controls the rate of increase of  $\nu$  approaching that point. Certainly one cannot pass this estimate in a naive way to all points in  $\mathcal{H}^{1,1}$ .

**Definition 4.4.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Fix  $\phi \in \overline{\mathcal{H}}$  and  $\tau > 0$ . Let

$$(4.2) \quad \mathcal{F}_{\phi, \tau}(\psi) = \frac{\overline{d}^2(\phi, \psi)}{2\tau} + \overline{\nu}(\psi).$$

Furthermore, set

$$(4.3) \quad \mu_{\phi, \tau} := \inf_{\psi \in \mathcal{H}} \mathcal{F}_{\phi, \tau}(\psi)$$

The quantity  $\mu$  is sometimes referred to as a *Moreau-Yosida approximation* of the given functional, in this case  $\nu$ . Finally, we define the *resolvent operator*

$$W_\tau : \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$$

by the property

$$\mathcal{F}_{\phi, \tau}(W_\tau(\phi)) = \mu_{\phi, \tau}.$$

The fact that there exists a unique minimizer for  $\mathcal{F}_{\phi, \tau}$  and so the map  $W_\tau$  is well defined will be shown later.

Using the resolvent operator we can define our notion of a discrete Calabi flow.

**Definition 4.5.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Given  $T > 0$ , consider a partition of  $[0, T]$ ,

$$0 = t_0 < t_1 < \cdots < t_m = T, \quad \tau_i = t_i - t_{i-1}.$$

We say that a sequence  $\{\phi_i\}_{i=0}^m \in \overline{\mathcal{H}}$  is a *discrete Calabi flow* with initial condition  $\phi_0$  if for all  $0 \leq i \leq m-1$ ,  $\phi_{i+1} = W_{\tau_i}(\phi_i)$ . We say that the solution has a *uniform step size*  $\tau$  if  $\tau_i = \tau$  for all  $i$ . Associated to any discrete Calabi flow is a one-parameter family  $\phi : [0, T] \rightarrow \overline{\mathcal{H}}$  where  $\phi|_{[t_i, t_{i+1})} = \phi_i$ .

**Definition 4.6.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. We say that a curve  $\phi : [0, T] \rightarrow \overline{\mathcal{H}}$  is a *K-energy minimizing movement* with initial condition  $\phi_0$  if there exists a sequence of partitions  $\{t_i^j\}_{i=0}^{p_j}$  with associated discrete Calabi flows  $\{\phi_i^j\}_{i=0}^{p_j}$  with initial condition  $\phi_0$  as in Definition 4.5 such that

- (1)  $\lim_{j \rightarrow \infty} \sup_i |\tau_i^j| = 0$ ,
- (2)  $\forall t \in [0, T], \phi^j(t) \rightarrow \phi(t)$ ,

where the convergence above is in the distance topology.

**Remark 4.7.** This definition allows for arbitrary step sizes, although the solutions we construct are convergent limits of discrete Calabi flows with uniform step sizes (see Theorem 5.5).

We proceed to derive some basic properties for  $\mathcal{F}$ . First we derive the first variation of  $\mathcal{F}$  at smooth points, and compute a further characterization of its critical points. To do this we need a preliminary lemma.

**Lemma 4.8.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Fix  $\phi \in \mathcal{H}$  and  $\psi \in C^\infty(M)$ . Then

$$\int_M \psi(s_\phi - \bar{s})\omega_\phi^n = \int_M \sqrt{-1}\partial\bar{\partial}\psi \wedge \left( -\log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^{n-1} \right) + \psi(\rho(\omega) - \bar{s}\omega_\phi) \wedge \omega_\phi^{n-1}.$$

*Proof.* We directly compute

$$\begin{aligned} \int_M \psi(s_\phi - \bar{s})\omega_\phi^n &= \int_M \psi(\rho(\omega_\phi) - \bar{s}\omega_\phi) \wedge \omega_\phi^{n-1} \\ &= \int_M \psi \left( \rho(\omega) - \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \frac{\omega_\phi^n}{\omega^n} - \bar{s}\omega_\phi \right) \wedge \omega_\phi^{n-1} \\ &= \int_M \sqrt{-1}\partial\bar{\partial}\psi \wedge \left( -\log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^{n-1} \right) + \psi(\rho(\omega) - \bar{s}\omega_\phi) \wedge \omega_\phi^{n-1}. \end{aligned}$$

□

**Lemma 4.9.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Fix  $\phi \in \mathcal{H}$  and  $\psi_s \in \mathcal{H}$  a one-parameter family of functions,  $s \in (-\epsilon, \epsilon)$ . Then

$$(4.4) \quad \left. \frac{d}{ds} \mathcal{F}_{\phi, \tau}(\psi_s) \right|_{s=0} = \left\langle \left\langle \frac{1}{\tau} \frac{\partial \gamma(1, 0)}{\partial t} - s_\psi + \bar{s}, \frac{d\psi}{ds} \right\rangle \right\rangle$$

where  $\gamma : [0, 1] \rightarrow \mathcal{H}$  is the unique  $C^{1,1}$  geodesic connecting  $\phi$  to  $\psi_0$ . Furthermore,  $\phi \in \mathcal{H}$  is a critical point for  $\mathcal{F}_{\phi, \tau}$  if and only if for all  $\eta \in C^\infty(M)$ ,

$$(4.5) \quad 0 = \int_M \left[ \eta \left( \frac{1}{\tau} \frac{\partial \gamma(1, 0)}{\partial t} + \bar{s} \right) \omega_\phi - \eta \rho(\omega) + \log \frac{\omega_\phi^n}{\omega^n} \sqrt{-1} \partial \bar{\partial} \eta \right] \wedge \omega_\phi^{n-1}.$$

*Proof.* Using Lemma 3.4 we compute the variation

$$\frac{d}{ds} \frac{d^2(\phi, \psi_s)}{2\tau} \Big|_{s=0} = \frac{d(\phi, \psi)}{\tau E(1, 0)^{\frac{1}{2}}} \left\langle \left\langle \frac{\partial \gamma(1, 0)}{\partial t}, \frac{d\psi}{ds} \right\rangle \right\rangle.$$

But since  $\gamma$  has constant speed, in particular we have  $E(1, 0)^{\frac{1}{2}} = d(\phi, \psi)$ . Combining this with the definition of  $K$ -energy and Lemma 4.8 yields the result.  $\square$

**Lemma 4.10.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold, and fix  $\phi \in \mathcal{H}$ ,  $\tau > 0$ . If  $\psi$  is a minimizer for  $\mathcal{F}_{\phi, \tau}$ , then  $I(\psi) = I(\phi)$ .*

*Proof.* Without loss of generality we assume  $\phi = 0$ , the general case being analogous. Let  $\psi_s = \psi + s$ . Certainly  $\psi_s$  is a geodesic. Moreover, it is clear by construction that  $\nu(\psi_s) = \nu(\psi)$  for all  $s$ . Since  $\psi$  realizes the minimum for  $\mathcal{F}_{\phi, \tau}$ , it thus follows that  $\psi$  realizes the minimum for  $d(\phi, \psi_s)$  in the variable  $s$ . Let  $\gamma$  denote the unique  $C^{1,1}$  geodesic connecting 0 to  $\phi$ . By Lemma 3.4, varying through the curve  $\psi_s$ , it follows that

$$0 = \left\langle \left\langle 1, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle_{\gamma} \Big|_{t=1} = \frac{d}{dt} I(\gamma_t) \Big|_{t=1}.$$

But since  $\gamma$  is a geodesic we conclude that

$$\frac{d^2}{dt^2} I(\gamma_t) = \frac{d}{dt} \left\langle \left\langle 1, \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle = \left\langle \left\langle 1, \frac{D}{dt} \frac{\partial \gamma}{\partial t} \right\rangle \right\rangle = 0.$$

Thus  $\frac{d}{dt} I(\gamma_t) = 0$  for all  $t$ , and hence  $I(\gamma_1) = I(\gamma_0)$ .  $\square$

## 5. LONG TIME EXISTENCE OF MINIMIZING MOVEMENTS

In this section we prove Theorem 1.3. As mentioned in the introduction, the proof will be an application of an theorem of Mayer [24] on the long time existence of minimizing movements for convex lower semicontinuous functionals on complete nonpositively curved metric spaces ([24] Theorem 1.13, Theorem 5.5 below). We begin by recalling the setup and statement of this theorem.

### 5.1. Mayer's Theorem.

**Definition 5.1.** A metric space  $(X, d)$  is a *path-length space* if any two points  $x, y \in X$  can be connected by a path  $\gamma : [0, 1] \rightarrow X$  such that for all  $t \in [0, 1]$ ,  $d(x, \gamma(t)) = td(x, y)$ . Such a path  $\gamma$  will be called a *constant speed geodesic*.

**Definition 5.2.** We say that a metric space  $(X, d)$  is an *NPC space* if  $(X, d)$  is a path-length space and for every choice of points  $a, b, c \in X$ , if  $\gamma : [0, 1] \rightarrow X$  denotes the unique geodesic path connecting  $b$  to  $c$ , then for all  $t \in [0, 1]$  one has

$$(5.1) \quad d(a, \gamma(t))^2 \leq (1-t)d(a, b)^2 + td(a, c)^2 - t(1-t)d(b, c)^2.$$



Inequality (5.1) is a kind of “hyperbolic triangle inequality” in that it holds on smooth manifolds with nonpositive curvature and intuitively says that triangles bend inwards. In §5.3 we will furthermore require the analogous “quadrilateral comparison” inequality for NPC spaces.

**Theorem 5.3.** ([21] *Corollary 2.1.3*) *Let  $(X, d)$  be a complete NPC space. Given  $x_0, x_1, y_0, y_1 \in X$ , let  $x_t$  and  $y_t$  denote the geodesics connecting  $x_0$  to  $x_1$  and  $y_0$  to  $y_1$  respectively. Then for all  $t \in [0, 1]$  one has*

$$d^2(x_t, y_0) + d^2(x_{1-t}, y_1) \leq d^2(x_0, y_0) + d^2(x_1, y_1) + 2t^2 d^2(x_0, x_1) \\ + t(d^2(y_0, y_1) - d^2(x_0, x_1)) - t(d(y_0, y_1) - d(x_0, x_1))^2.$$

**Definition 5.4.** Let  $(X, d)$  be an NPC space. Given  $B \geq 0$ , a function  $f : X \rightarrow \mathbb{R}$  is  $B$ -convex if for all  $x_0, x_1 \in X$ , we let  $x_t : [0, 1] \rightarrow X$  denotes the geodesic connecting  $x_0$  to  $x_1$ , then one has

$$(5.2) \quad f(x_t) \leq (1-t)f(x_0) + tf(x_1) + Bt(1-t)d^2(x_0, x_1).$$

for all  $t \in [0, 1]$ .

**Theorem 5.5.** ([24] *Theorem 1.13*) *Let  $(X, d)$  be a complete NPC space, and let  $f : X \rightarrow (-\infty, \infty]$  satisfy*

- (1)  $f$  is lower semicontinuous,
- (2)  $f$  is  $B$ -convex for some  $B \geq 0$ .

Fix  $y \in X$  and let

$$A := -\min \left\{ 0, \liminf_{d(x,y) \rightarrow \infty} \frac{f(x)}{d^2(x, y)} \right\}, \\ I_A := \begin{cases} (0, \infty) & \text{for } A = 0, \\ (0, \frac{1}{16A}] & \text{for } A > 0. \end{cases}$$

Then given  $x_0 \in X$  with  $f(x_0) < \infty$ , there exists a function  $x : I_A \rightarrow X$  satisfying

$$(5.3) \quad x(t) = \lim_{n \rightarrow \infty} W_{\frac{t}{n}}^n(x_0),$$

$$(5.4) \quad f(x(t)) \leq f(x_0) \text{ for all } t \in I_A,$$

$$(5.5) \quad \lim_{t \rightarrow 0} x(t) = x_0.$$

Furthermore, the convergence in (5.3) is uniform on compact subintervals of  $I_A$ .

**Remark 5.6.** The operators  $W_{\frac{t}{n}}^n$  are iterations of the resolvent operator as in Definition 4.4. Note that the theorem first of all asserts the long time existence of discrete gradient flows, in the sense of Definition 4.4, with any initial condition and arbitrarily small uniform step size. Moreover, it asserts that a sequence of such converges in the distance topology to some limiting path through  $X$ , which by definition is a minimizing movement.

**Remark 5.7.** Of course implicit in the statement of the theorem is that for every  $t \in [0, \infty)$ , the sequence  $\{W_{\frac{t}{n}}^n(x_0)\}$  is Cauchy in  $(X, d)$ , and so in particular lies in a

ball of some controlled size around 0. However, it is relevant to the proof of Theorem 1.5 to precisely exhibit the dependence of this distance on the initial data. This is included in the work of Mayer, and for convenience we include the technical lemma below. Moreover, this lemma makes clear the role of the constant  $A$ , which as a measure of the decay rate of  $f$  in turn controls how fast the distance of points along the flow can grow.

**Lemma 5.8.** ([24] Lemma 1.11) *Let  $(X, d)$  be a complete NPC space, and let  $f : X \rightarrow (-\infty, \infty]$  satisfy the conditions of Theorem 5.5. Let  $x_0 \in X$  satisfy  $f(x_0) < \infty$ , and let  $x_{j+1} = W_h(x_j)$ . If  $A$  as defined in Theorem 5.5 satisfies  $A = 0$ , then for  $T > 0$ , one has*

$$d^2(x_0, x_j) \leq Bjh,$$

where  $B$  depends on  $f(x_0), d(x_0, y), T$ , and  $\inf_{B_{R(T)}(y)} f(x)$ , where  $R_T$  is chosen so that for all  $x$  satisfying  $d(x, y) \geq R$ , one has  $f(x) \geq -\frac{d^2(x, y)}{8T}$ .

*Proof.* By the definition of the resolvent operator, we have that  $\mathcal{F}_{x_i, h}(x_{i+1}) \leq \mathcal{F}_{x_i, h}(x_i)$ , or in other words

$$(5.6) \quad \frac{1}{2h} d^2(x_{i+1}, x_i) \leq f(x_i) - f(x_{i+1}).$$

Using the triangle inequality and Cauchy-Schwarz yields

$$(5.7) \quad d^2(x_0, x_j) \leq \left( \sum_{i=0}^{j-1} d(x_i, x_{i+1}) \right)^2 \leq j \sum_{i=0}^{j-1} d^2(x_i, x_{i+1}).$$

Combining (5.6) and (5.7) yields

$$d^2(x_0, x_j) \leq 2jh(f(x_0) - f(x_j)).$$

Now choose  $K > 0$  so that for all  $x \in X$  one has

$$f(x) \geq -K - \frac{d^2(x, y)}{8T}.$$

Note that  $K$  can be chosen to be  $K = \min \{1, -\inf_{B_R(y)} f\}$  where  $R$  is chosen so that for all  $x$  satisfying  $d(x, y) \geq R$ , one has  $f(x) \geq -\frac{d^2(x, y)}{8T}$ , which exists by our hypothesis on  $A$ . Thus we have

$$d^2(x_0, x_j) \leq 2jh \left( f(x_0) + K + \frac{1}{4T} (d^2(x_j, x_0) + d^2(x_0, y)) \right).$$

Since  $jh \leq T$  we conclude that

$$d^2(x_0, x_j) \leq 4jh \left( f(x_0) + K + \frac{1}{4T} d^2(x_0, y) \right).$$

□

## 5.2. Proof of Theorem 1.3.

## 5.2.1. NPC Property.

**Lemma 5.9.** *Let  $(X, d)$  be a metric space such that every pair of points  $x, y \in X$  is connected by a unique geodesic  $\gamma : [0, 1] \rightarrow (\overline{X}, \overline{d})$ , and that the hyperbolic triangle inequality (3.4) holds for triples of points  $a, b, c \in X$ . Then  $(\overline{X}, \overline{d})$  is an NPC space.*

*Proof.* The proof is summarized in Figure 1. Fix  $\overline{x}, \overline{y}$  two points in  $\overline{X}$ , represented by Cauchy sequences  $\{x_n\}, \{y_n\}$ . Let  $\gamma_n : [0, 1] \rightarrow \overline{X}$  denote the unique constant speed geodesic connecting  $x_n$  to  $y_n$ . In particular, one has  $\overline{d}(x_n, \gamma_n(s)) = s\overline{d}(x_n, y_n)$  for all  $n \in \mathbb{N}, s \in [0, 1]$ . Let  $l_n := \overline{d}(x_n, y_n)$ . By the definition of the distance function on the completion one has that  $\lim_{n \rightarrow \infty} l_n = \overline{d}(x, y) = l$ . Fix some  $s \in (0, 1)$ . We claim that  $\{\gamma_n(s)\}$  is a Cauchy sequence. Given  $\epsilon > 0$ , first choose  $N$  large so that for all  $n, m > N$ ,  $\overline{d}(x_m, x_n) < \epsilon$ ,  $\overline{d}(y_m, y_n) < \epsilon$ . Furthermore, fix a point  $\tilde{\gamma}_n(s) \in X$  such that  $d(\gamma_n(s), \tilde{\gamma}_n(s)) < \epsilon$ . Now consider the triangle  $\Delta abc$  with  $a = \tilde{\gamma}_n(s), b = x_m, c = y_m$ . We may apply (3.4) to yield

$$\overline{d}(\tilde{\gamma}_n(s), \gamma_m(s))^2 \leq (1-s)d(\tilde{\gamma}_n(s), x_m)^2 + sd(\tilde{\gamma}_n(s), y_m)^2 - s(1-s)d(x_m, y_m)^2.$$

But now note

$$\begin{aligned} d(\tilde{\gamma}_n(s), x_m) &\leq \overline{d}(\tilde{\gamma}_n(s), \gamma_n(s)) + \overline{d}(\gamma_n(s), x_n) + \overline{d}(x_n, x_m) \\ &\leq sl_n + 2\epsilon. \end{aligned}$$

Likewise

$$d(\tilde{\gamma}_n(s), y_m) \leq (1-s)l_n + 2\epsilon.$$

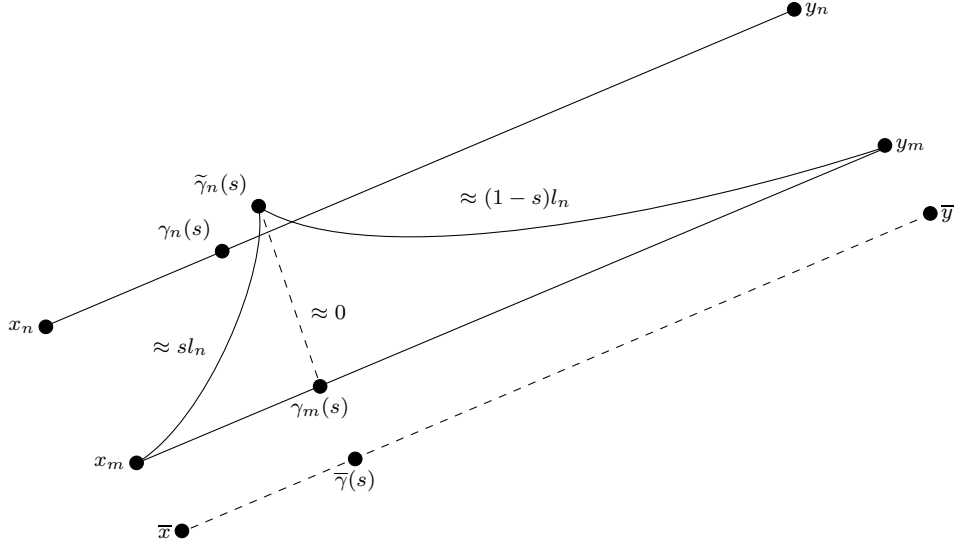


FIGURE 1. Convergence of geodesics in  $\overline{\mathcal{H}}$

Combining these facts yields

$$\begin{aligned}
\bar{d}(\gamma_n(s), \gamma_m(s))^2 &\leq \bar{d}(\gamma_n(s), \tilde{\gamma}_n(s))^2 + \bar{d}(\tilde{\gamma}_n(s), \gamma_m(s))^2 \\
&\leq (1-s)(sl_n + \epsilon)^2 + s((1-s)l_n + \epsilon)^2 - s(1-s)l_m^2 + C(s, l)\epsilon \\
&= s(1-s)(sl_n^2 + (1-s)l_n^2 - l_m^2) + C(s, l)\epsilon \\
&= s(1-s)(l_n^2 - l_m^2) + C(s, l)\epsilon.
\end{aligned}$$

Since  $\lim_{n \rightarrow \infty} l_n = l$ , the claim follows. In particular, since  $\{\gamma_n(s)\}$  is Cauchy for every  $s$ , this defines the limiting curve  $\bar{\gamma}$ . It is clear that the property  $\bar{d}(\bar{x}, \bar{\gamma}(s)) = sl$  passes to the limit. Furthermore, given that we have established that geodesics connecting Cauchy sequences of points in  $X$  converge to a limiting geodesic curve connecting two points in  $\bar{X}$ , property (3.4) certainly passes to the limit as well. Also, uniqueness of the limiting geodesic follows by a very similar argument to the existence. In particular if  $\tilde{\gamma} : [0, 1] \rightarrow \bar{X}$  is another curve satisfying  $d(x, \tilde{\gamma}(s)) = sd(x, y)$  and  $d(\tilde{\gamma}(s), y) = (1-s)d(x, y)$  for all  $s$ , then by using the hyperbolic triangle inequality for the triangle  $\Delta xy\tilde{\gamma}(s)$  one obtains directly that  $\tilde{\gamma}$  is the same as the curve constructed above.  $\square$

### 5.2.2. Lower semicontinuity.

**Lemma 5.10.** *The function  $\bar{f}$  as defined in (4.1) is lower semicontinuous.*

*Proof.* Fix  $x_\infty \in \bar{X}$ , and choose a sequence  $\{x_i \in \bar{X}\}$  converging to  $x_\infty$  in the distance topology. By a diagonalization argument, for each  $x_i$  we may choose a sequence  $x_i^j \in X$  such that

$$\begin{aligned}
\lim_{j \rightarrow \infty} d(x_i^j, x_i) &= 0 \\
\lim_{j \rightarrow \infty} f(x_i^j) &= \bar{f}(x_i).
\end{aligned}$$

Note that if  $x_i \in X$  one simply chooses  $x_i^j = x_i$  for all  $j$ . For each  $i$  choose  $N_i$  such that for all  $j \geq N_i$ ,  $d(x_i^j, x_i) \leq \frac{1}{i}$  and  $f(x_i^j) \leq \bar{f}(x_i) + \frac{1}{i}$ . It follows from the triangle inequality that  $\lim_{i \rightarrow \infty} d(x_i^{N_i}, x_\infty) = 0$ . If  $x_\infty \in X$  then since  $f$  is lower semicontinuous on  $X$  one has

$$\bar{f}(x_\infty) = f(x_\infty) \leq \lim_{i \rightarrow \infty} f(x_i^{N_i}) \leq \lim_{i \rightarrow \infty} \bar{f}(x_i) + \frac{1}{i} = \lim_{i \rightarrow \infty} \bar{f}(x_i).$$

On the other hand, if  $x_\infty \in \bar{X} \setminus X$  then by definition we have

$$\bar{f}(x_\infty) \leq \lim_{i \rightarrow \infty} f(x_i^{N_i}) \leq \lim_{i \rightarrow \infty} \bar{f}(x_i) + \frac{1}{i} = \lim_{i \rightarrow \infty} \bar{f}(x_i).$$

The lemma follows.  $\square$

**Lemma 5.11.** *The function  $\nu$  is lower semicontinuous on  $(\mathcal{H}, d)$ .*

*Proof.* This follows directly from Theorem 3.12.  $\square$

**Lemma 5.12.** *For every  $\phi \in \bar{\mathcal{H}}$ ,  $\bar{\nu}(\phi) > -\infty$ .*

*Proof.* Fix  $\phi \in \overline{\mathcal{H}}$ , and  $\{\phi_i\} \in \mathcal{H}$  any sequence converging to  $\phi$  in the distance topology. By the triangle inequality we have, for sufficiently large  $i$ ,

$$d(0, \phi_i) \leq d(0, \phi) + d(\phi, \phi_i) \leq C.$$

Thus by Theorem 3.12 there exists a constant  $C$  depending only on  $\phi$  such that

$$\lim_{i \rightarrow \infty} \nu(\phi_i) \geq -C.$$

We conclude that  $\overline{\nu}(\phi) > -\infty$ .  $\square$

**5.2.3. Convexity.** In this subsection we establish geodesic convexity of  $\overline{\nu}$  on  $\overline{\mathcal{H}}$ . We begin with some preliminary lemmas.

**Lemma 5.13.** *Given  $\phi \in \mathcal{H}^{1,1}$  there exists a sequence  $\{\phi_i\} \in \mathcal{H}$  converging to  $\phi$  in the weak  $C^{1,1}$  topology such that*

$$\lim_{i \rightarrow \infty} \nu(\phi_i) = \nu(\phi).$$

*Proof.* By definition of closure in the weak  $C^{1,1}$  topology, given  $\phi \in \mathcal{H}^{1,1}$  there exists a sequence  $\{\phi_i\} \in \mathcal{H}$  such that

- (1)  $|\phi_i|_{C^{1,1}} \leq A$  for all  $i$
- (2)  $\phi_i \rightarrow \phi$  weakly in  $W^{2,p}$  for a given large constant  $p$ .
- (3)  $\phi_i \rightarrow \phi$  strongly in  $C^{1,\alpha}$  for  $0 < \alpha < 1$ .

By the strong convergence in  $C^{1,\alpha}$  it follows directly from (2.3) and (2.7) that the  $I$  and  $J$  terms appearing in (2.10) converge to the limiting value at  $\phi$ . To deal with the remaining term, first we claim that  $\frac{\omega_{\phi_i}^n}{\omega^n} \rightarrow \frac{\omega_\phi^n}{\omega^n}$  weakly in  $L^p$ . To show this, fix  $\psi \in L^q$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and note that

$$\begin{aligned} \lim_{i \rightarrow \infty} \left| \int_M \psi \left( \frac{\omega_{\phi_i}^n}{\omega^n} - \frac{\omega_\phi^n}{\omega^n} \right) \omega^n \right| &= \lim_{i \rightarrow \infty} \left| \int_M \psi (\omega_{\phi_i}^n - \omega_\phi^n) \right| \\ &= \lim_{i \rightarrow \infty} \left| \int_M \psi \left( \sqrt{-1} \partial \bar{\partial} (\phi_i - \phi) \wedge \left( \sum_{j=0}^{n-1} \omega_{\phi_i}^j \wedge \omega_\phi^{n-1-j} \right) \right) \right| \\ &\leq \lim_{i \rightarrow \infty} \int_M |\psi| |\partial \bar{\partial} (\phi_i - \phi)|_\omega (1 + A)^{n-1} \omega^n \\ &= 0 \end{aligned}$$

by the weak convergence in  $W^{2,p}$ . This implies  $\frac{\omega_{\phi_i}^n}{\omega^n} \rightarrow \frac{\omega_\phi^n}{\omega^n}$  strongly in  $L^{p'}, p' < p$ , which in turn implies that there exists a subsequence of  $\{\phi_n\}$  such that the volume form ratios converge almost everywhere. This implies that  $f_n := \frac{\omega_{\phi_i}^n}{\omega^n} \log \frac{\omega_{\phi_i}^n}{\omega^n}$  converges almost everywhere to  $f := \frac{\omega_\phi^n}{\omega^n} \log \frac{\omega_\phi^n}{\omega^n}$ . By a standard measure theoretic lemma, since  $M$  is compact, and in particular the measure induced by  $\omega$  is finite, we may choose a further subsequence to obtain  $f_n \rightarrow f$  almost everywhere and with respect to measure. Note that the sequence  $\{f_n\}$  satisfies uniform upper and lower bounds depending on  $A$ . By the convergence in measure, for any  $\epsilon > 0$  we may choose  $N_\epsilon$  large so that for

all  $n \geq N$ ,  $U_n := \{x \in M \mid |f_n - f| \geq \epsilon\}$  satisfies  $\int_{U_n} \omega^n \leq \epsilon$ . We then have, for all  $n \geq N_\epsilon$ ,

$$\begin{aligned} \left| \int_M (f - f_n) \omega^n \right| &\leq \int_M |f - f_n| \omega^n \\ &= \int_{U_n} |f - f_n| \omega^n + \int_{M \setminus U_n} |f - f_n| \omega^n \\ &\leq 2C(A)\epsilon + V\epsilon. \end{aligned}$$

It follows that the sequence  $\{\phi_{N_{\frac{1}{n}}}\}$  satisfies the required properties.  $\square$

**Lemma 5.14.** *Given  $\{\phi_i\} \in \mathcal{H}^{1,1}$  such that  $\{\phi_i\} \rightarrow \phi \in \mathcal{H}^{1,1}$  in  $C^{1,\alpha}$ , one has*

$$\overline{\nu}(\phi) \leq \liminf_{i \rightarrow \infty} \nu(\phi_i).$$

*Proof.* By Lemma 5.13 we may choose sequences  $\{\phi_i^j\} \in \mathcal{H}$  converging to  $\phi_i$  in the weak  $C^{1,1}$  topology such that

$$\lim_{j \rightarrow \infty} \nu(\phi_i^j) = \nu(\phi_i).$$

Since convergence in the weak  $C^{1,1}$  topology implies convergence in  $C^{1,\alpha}$ , which implies convergence in the distance topology, a simple diagonalization argument yields a sequence  $\phi_i^{j_i}$  converging to  $\phi$  in the distance topology, such that

$$\lim_{i \rightarrow \infty} \nu(\phi_i^{j_i}) = \liminf_{i \rightarrow \infty} \nu(\phi_i).$$

The result follows from the definition of  $\overline{\nu}$ .  $\square$

**Lemma 5.15.** *Given the setup of Lemma 5.9, let  $f : X \rightarrow \mathbb{R}$  be lower semicontinuous. If  $\overline{f}$  is convex along geodesics connecting points in  $X$ , then  $\overline{f} : \overline{X} \rightarrow \mathbb{R}$  is geodesically convex.*

*Proof.* Fix  $\overline{x}, \overline{y} \in \overline{X}$ , and fix  $x_n \rightarrow \overline{x}$  a Cauchy sequence in  $X$  such that

$$\lim_{n \rightarrow \infty} f(x_n) = \overline{f}(\overline{x}).$$

Likewise define  $y_n$ . These sequences exist by the definition of  $\overline{f}$ . Let  $\gamma_n : [0, 1] \rightarrow \overline{X}$  denote the unique geodesic connecting  $x_n$  to  $y_n$ . By the assumed convexity of  $\overline{f}$  on geodesics connecting points in  $X$  one obtains

$$\overline{f}(\gamma_n(t)) \leq (1-t)\overline{f}(x_n) + t\overline{f}(y_n).$$

Taking the limit yields

$$\lim_{n \rightarrow \infty} \overline{f}(\gamma_n(t)) \leq (1-t)\overline{f}(\overline{x}) + t\overline{f}(\overline{y}).$$

But as in the proof of Lemma 5.9, we know that  $\lim_{n \rightarrow \infty} \gamma_n(t) = \overline{\gamma}(t)$ , where  $\overline{\gamma} : [0, 1] \rightarrow \overline{X}$  denotes the geodesic connecting  $\overline{x}$  to  $\overline{y}$ . Therefore by lower semicontinuity of  $\overline{f}$  we obtain

$$\overline{f}(\overline{\gamma}(t)) \leq \lim_{n \rightarrow \infty} \overline{f}(\gamma_n(t)) \leq (1-t)\overline{f}(\overline{x}) + t\overline{f}(\overline{y}).$$

Thus  $\overline{f}$  is geodesically convex.  $\square$

**Proposition 5.16.**  $\overline{\nu} : \overline{\mathcal{H}} \rightarrow \mathbb{R}$  is geodesically convex, that is, given  $\phi : [0, 1] \rightarrow \overline{\mathcal{H}}$  a geodesic, for all  $t \in [0, 1]$  one has

$$(5.8) \quad \overline{\nu}(\phi_t) \leq (1-t)\overline{\nu}(\phi_0) + t\overline{\nu}(\phi_1).$$

*Proof.* The main step is to show that (5.8) holds for  $\phi_0, \phi_1 \in \mathcal{H}$ , in which case  $\phi_t \in \mathcal{H}^{1,1}$  is the unique  $C^{1,1}$  geodesic connecting  $\phi_0$  to  $\phi_1$ . The proposition then follows from Lemma 5.15. We prove this fact using the “oval approximations” of  $C^{1,1}$  geodesics as introduced in [11]. Consider the surface with boundary in the plane  $\mathbb{C}$ ,

$$\Sigma^l := [-l, l] \times [0, 1] \cup D_{\frac{1}{2}}(-l, \frac{1}{2}) \cup D_{\frac{1}{2}}(l, \frac{1}{2})$$

where  $D_r(s, t)$  denotes the Euclidean disc of radius  $r$  around the point  $(s, t)$ . The set  $\Sigma^l$  is a “stadium” shape, whose boundary consists of the two lines  $(-l, l) \times \{0\}$ ,  $(-l, l) \times \{1\}$  and two semicircles denoted  $C_{\pm}$ . Also we set  $\Sigma^{\infty} := (-\infty, \infty) \times [0, 1]$ . Fix diffeomorphisms  $\gamma_{\pm} : [0, 1] \rightarrow C_{\pm}$  such that  $\gamma_{\pm}(0) = (\pm l, 0)$ ,  $\gamma_{\pm}(1) = (\pm l, 1)$ . Given  $\phi_0, \phi_1 \in \mathcal{H}$  consider the map

$$\begin{aligned} \phi^l : \Sigma^l &\rightarrow \mathcal{H} \\ &: (0, t) \rightarrow \phi_0 \\ &: (1, t) \rightarrow \phi_1 \\ &: (s, t) \rightarrow (\gamma_{\pm}^{-1}(s, t)) \psi_1 + (1 - \gamma_{\pm}^{-1}(s, t)) \psi_0 \quad \text{for } (s, t) \in C_{\pm}. \end{aligned}$$

In particular, the boundary value on the two semicircles smoothly interpolates between  $\phi_0$  and  $\phi_1$ . Let  $\psi^l : \partial\Sigma^l \rightarrow \mathcal{H}$  denote the boundary map of  $\phi^l$ . Furthermore, for any  $\delta > 0$ , by Theorem 3.21 there exists a boundary map  $\psi^{l,\delta} : \partial\Sigma^l \rightarrow \mathcal{H}$  and an almost smooth geodesic  $\phi^{l,\delta} : \Sigma^l \rightarrow \mathcal{H}$  with this boundary condition, such that

$$|\phi^{l,\delta}|_{C^{1,1}} \leq A, \quad \lim_{\delta \rightarrow 0} \max_{\partial\Sigma^l \times M} \|\psi^{l,\delta} - \psi^l\|_{C^{2,\alpha}} = 0.$$

The constant  $A$  above is independent of both  $l$  and  $\delta$ , so in particular the  $K$ -energies in the image of  $\phi^{l,\delta}$  are uniformly bounded as well. Let  $\nu^{l,\delta}(s, t) := \nu(\phi^{l,\delta}(s, t))$ . By Theorem 3.22  $\nu^{l,\delta}$  is weakly subharmonic. Furthermore let  $f^{l,\delta}(s, t) = \nu^{l,\delta}(s, t) - (1-t)\nu(\phi_0) - t\nu(\phi_1)$ . Certainly  $f^{l,\delta}$  is also weakly subharmonic.

Now fix  $\kappa : \mathbb{R} \rightarrow \mathbb{R}$  a smooth nonnegative cutoff function such that

$$\kappa \equiv 1 \text{ on } [-\frac{1}{2}, \frac{1}{2}], \quad \text{supp } \kappa \subset [-\frac{3}{4}, \frac{3}{4}].$$

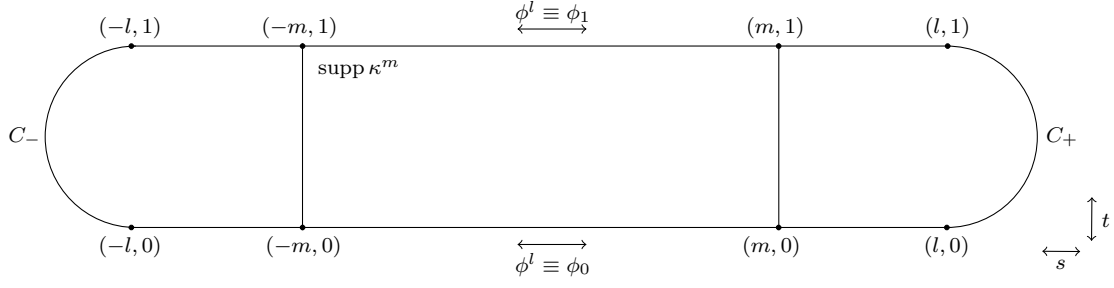
Furthermore set

$$\kappa^m(s) = \frac{\kappa(\frac{s}{m})}{\overline{\kappa}}, \quad \overline{\kappa} := \int_{-\infty}^{\infty} \kappa(s) ds.$$

The set up is summarized in Figure 2.

For  $\epsilon > 0$  let  $F^{l,\delta,\epsilon}$  denote a smooth subharmonic function on  $\Sigma^l$  such that

$$\lim_{\epsilon \rightarrow 0} F^{l,\delta,\epsilon} = f^{l,\delta} =: F^{l,\delta,0}$$

FIGURE 2. Setup of  $\Sigma_l$ 

uniformly in  $C^1(\Sigma^l)$ . This can be achieved by extending  $f^{l,\delta}$  continuously from the boundary and using mollifiers. Lastly, set

$$W^{l,m,\delta,\epsilon}(t) := \int_{-\infty}^{\infty} \kappa^m(s) F^{l,\delta,\epsilon}(s, t) ds.$$

Certainly by construction  $W^{l,m,\delta,\epsilon}$  is a  $C^{2,\alpha}$  function of  $t$ , and so we compute

$$\begin{aligned} \frac{d^2 W^{l,m,\delta,\epsilon}}{dt^2} &= \int_{-\infty}^{\infty} \kappa^m(s) \frac{\partial^2 F^{l,\delta,\epsilon}}{\partial t^2}(s, t) ds \\ &= \int_{-\infty}^{\infty} \kappa^m(s) \left[ \Delta_{s,t} F^{l,\delta,\epsilon} - \frac{\partial^2 F^{l,\delta,\epsilon}}{\partial s^2} \right](s, t) ds \\ &\geq - \int_{-\infty}^{\infty} \kappa^m(s) \left[ \frac{\partial^2 F^{l,\delta}}{\partial s^2}(s, t) \right] ds \\ &= - \int_{-\infty}^{\infty} \frac{d^2 \kappa^m}{ds^2} F^{l,\delta,\epsilon}(s, t) ds \\ &= - \frac{1}{\bar{\kappa} m^2} \int_{-\infty}^{\infty} \frac{d^2 \kappa}{ds^2} \left( \frac{s}{m} \right) F^{l,\delta,\epsilon}(s, t) ds. \end{aligned}$$

As noted above  $|f^{l,\delta}(s, t)|$  has a bound independent of  $l, \delta, s, t$ , and thus  $F^{l,\delta,\epsilon}$  has a uniform bound independent of  $l, \delta, \epsilon$ . Thus

$$\begin{aligned} \frac{d^2 W^{l,m,\delta,\epsilon}}{dt^2} &\geq - \frac{C}{\bar{\kappa} m^2} \int_{-\infty}^{\infty} \left| \frac{d^2 \kappa}{ds^2} \right| \left( \frac{s}{m} \right) ds \\ &= - \frac{C}{\bar{\kappa} m} \int_{-\infty}^{\infty} \left| \frac{d^2 \kappa}{ds^2} \right| (s) ds \\ &\geq - \frac{C}{m}. \end{aligned}$$

With this estimate on the second derivative and the fact that the boundary values are  $o(\delta) + o(\epsilon)$ , by elementary calculus arguments we conclude that, for any  $t \in [0, 1]$ ,

$$W^{l,m,\delta,\epsilon}(t) \leq \frac{C}{m} + o(\delta) + o(\epsilon).$$



Due to the uniform convergence of  $F^{l,\delta,\epsilon}$  to  $F^{l,\delta,0}$  as  $\epsilon \rightarrow 0$  we may send  $\epsilon$  to 0 in this estimate to yield

$$(5.9) \quad W^{l,m,\delta,0}(t) \leq \frac{C}{m} + o(\delta).$$

Since the  $\phi^{l,\delta}$  satisfy a uniform  $C^{1,1}$  bound, and moreover solutions to the boundary value problem are unique, it follows that

$$\sup_{\Sigma^l \times M} \|\phi^l - \phi^{l,\delta}\|_{C^{1,\alpha}} = o(\delta).$$

In particular this convergence implies convergence in the distance topology. Furthermore, by ([7] Proposition 4.6),  $\phi^l$  converges to  $\phi^0$  in  $C^{1,\alpha}$  on any fixed compact subset of  $\Sigma^\infty \times M$ . In particular, this convergence holds on  $\Sigma^m \times M$  for any given  $m$ . Thus, given arbitrary  $\gamma > 0$  we may choose  $l$  sufficiently large and  $\delta$  sufficiently small so that for all  $s, t \in \Sigma^m$ ,  $d(\phi^{l,\delta}(s, t), \phi^0(t)) \leq \gamma$ . Therefore by Lemma 5.14 we obtain for these choices and all  $s, t \in \Sigma^m$ ,

$$\begin{aligned} f^{l,\delta}(s, t) &= \nu^{l,\delta}(s, t) - (1-t)\nu(\phi_0) - t\nu(\phi_1) \\ &\geq \bar{\nu}(\phi^0(t)) - o(\gamma) - (1-t)\nu(\phi_0) - t\nu(\phi_1). \end{aligned}$$

By the definition of  $W$  this implies

$$W^{l,m,\delta,0} \geq \bar{\nu}(\phi^0(t)) - o(\gamma) - (1-t)\nu(\phi_0) - t\nu(\phi_1).$$

Combining this with (5.9) yields

$$\bar{\nu}(\phi^0(t)) \leq (1-t)\nu(\phi_0) + t\nu(\phi_1) + o(\gamma) + o(\delta) + \frac{C}{m}.$$

Due to the uniform convergence discussed above on  $\Sigma^m$ , we first send  $l$  to infinity and  $\delta$  to 0 (which implies  $\gamma \rightarrow 0$  as discussed above) to yield

$$\bar{\nu}(\phi^0(t)) \leq (1-t)\nu(\phi_0) + t\nu(\phi_1) + \frac{C}{m} = (1-t)\bar{\nu}(\phi_0) + t\bar{\nu}(\phi_1) + \frac{C}{m}.$$

Taking the limit as  $m \rightarrow \infty$  yields the result.  $\square$

#### 5.2.4. Main Proof.

*Proof of Theorem 1.3.* Fix  $(M^{2n}, \omega, J)$  a compact Kähler manifold. By Lemmas 3.11 and 5.9 it follows that  $(\bar{\mathcal{H}}, \bar{d})$  is an NPC space. Furthermore, by Lemmas 5.10 and 5.11 the function  $\bar{\nu} : \bar{\mathcal{H}} \rightarrow \mathbb{R}$  is lower semicontinuous. By Proposition 5.16  $\bar{\nu}$  is geodesically convex. Fix  $\phi \in \mathcal{H}$ . By Theorem 3.12 we conclude that

$$\liminf_{\psi \in \mathcal{H}, d(\phi, \psi) \rightarrow \infty} \frac{\nu(\psi)}{d(\phi, \psi)^2} \geq \liminf_{\psi \in \mathcal{H}, d(\phi, \psi) \rightarrow \infty} \frac{\nu(\phi) - d(\phi, \psi)\sqrt{\mathcal{C}(\phi)}}{d(\phi, \psi)^2} = 0.$$

By definition this inequality passes to  $\bar{\nu}$ , and therefore in the notation of Theorem 5.5 we have shown  $A = 0$ . The theorem follows from Theorem 5.5.  $\square$

**5.3. Further properties of minimizing movements.** The theory of minimizing movements comes with a host of a priori regularity results which seek to exhibit the manner in which these can be thought of as gradient flows. We record some of these results here as immediate corollaries of results in [24]. First, one can further characterize the paths of Theorem 5.5 as curves of “steepest descent.”

**Definition 5.17.** Given  $(X, d)$  a complete NPC space and  $f : X \rightarrow \mathbb{R}$  a lower semi-continuous function, for  $x \in X$  let

$$|\nabla_- f|(x) = \max \left\{ \limsup_{y \rightarrow x} \frac{f(x) - f(y)}{d(x, y)}, 0 \right\}.$$

**Theorem 5.18.** ([24] Theorem 2.14) *Given the setup of Theorem 5.5, if  $x(t_0)$  is not a stationary point for  $f$ , then*

$$\lim_{t \rightarrow t_0^+} \frac{f(x(t_0)) - f(x(t))}{d(x(t), x(t_0))} = |\nabla_- f|(x(t_0)).$$

Moreover, for  $t_0 > 0$  this limit is finite.

**Theorem 5.19.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Let  $u_t : [0, \infty) \rightarrow \overline{\mathcal{H}}$  be a  $K$ -energy minimizing movement as in Theorem 1.3. Then*

$$\lim_{s \rightarrow 0^+} \frac{d(u_{t+s}, u_t)}{s} = |\nabla_- \overline{\nu}|(u_t).$$

*Proof.* This is an immediate corollary of ([24] Theorem 2.17) □

**Theorem 5.20.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Let  $u_t : [0, \infty) \rightarrow \overline{\mathcal{H}}$  be a  $K$ -energy minimizing movement as in Theorem 1.3. Then for almost all  $t > 0$  one has*

$$\frac{d\overline{\nu}(u_t)}{dt} = -|\nabla_- \overline{\nu}|^2(u_t).$$

*Proof.* This is an immediate corollary of ([24] Corollary 2.18) □

The construction of minimizing movements also allows one to derive interesting properties of the flow map on  $\overline{\mathcal{H}}$ .

**Definition 5.21.** Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. The  $K$ -energy flow map is

$$\begin{aligned} F : \overline{\mathcal{H}} \times [0, \infty) &\rightarrow \overline{\mathcal{H}} \\ &: (u_0, t) \rightarrow u_t, \end{aligned}$$

where  $u_t$  is the minimizing movement with initial condition  $u_0$  guaranteed by Theorem 1.3.

**Theorem 5.22.** ([24] Theorem 2.2, Corollary 2.3, Theorem 2.5, Corollary 2.6)

- (1) *Given  $u_0 \in \overline{\mathcal{H}}$  with  $\overline{\nu}(u_0) < \infty$ , the map  $t \rightarrow F_t(u_0)$  is uniformly Hölder continuous of exponent  $\frac{1}{2}$ . More specifically, there exists a constant  $C$  such that for any  $0 \leq s \leq t$ , one has*

$$d(F_s(u_0), F_t(u_0)) \leq C(t - s)^{\frac{1}{2}}.$$

- (2) The map  $F : \{\bar{\phi} \in \overline{\mathcal{H}} | \bar{\nu}(\bar{\phi}) < \infty\} \times [0, \infty) \rightarrow \overline{\mathcal{H}}$  is continuous.
- (3)  $F$  satisfies the semigroup property, i.e. for  $s, t \geq 0$  one has  $F_{s+t} = F_s \circ F_t$ .
- (4) Given  $u_0 \in \overline{\mathcal{H}}$  with  $\bar{\nu}(u_0) < \infty$ , the map  $t \rightarrow \bar{\nu}(F_t(u_0))$  is nonincreasing.

It was shown by Calabi-Chen that the distance between two points in  $\mathcal{H}$  decreases when each is flowed along Calabi flow. We reproduce this property for minimizing movements by adapting an argument from [24].

*Proof of Theorem 1.4.* The proof is adapted from ([24] Lemma 1.12) We will show that the resolvent operator is distance nonincreasing, and this implies the theorem due to the nature of the convergence in Theorem 1.3. Fix  $\psi_0, \psi_1 \in \mathcal{H}$ ,  $\tau > 0$ , and let  $\phi_0 = W_\tau(\psi_0), \phi_1 = W_\tau(\psi_1)$ . Let  $\phi_t$  denote the geodesic connecting  $\phi_0$  to  $\phi_1$ . By using the quadrilateral comparison inequality for NPC spaces (Theorem 5.3), one has

$$\begin{aligned} \frac{1}{2\tau} d^2(\phi_t, \psi_0) + \frac{1}{2\tau} d^2(\phi_{1-t}, \psi_1) &\leq \frac{1}{2\tau} [d^2(\phi_0, \psi_0) + d^2(\phi_1, \psi_1) + 2t^2 d^2(\phi_0, \phi_1) \\ &\quad + t(d^2(\psi_0, \psi_1) - d^2(\phi_0, \phi_1)) - t(d(\psi_0, \psi_1) - d(\phi_0, \phi_1))^2] \end{aligned}$$

Note also that the convexity of  $\bar{\nu}$  implies that

$$\bar{\nu}(\phi_t) + \bar{\nu}(\phi_{1-t}) \leq \bar{\nu}(\phi_0) + \bar{\nu}(\phi_1).$$

Combining these two inequalities yields

$$\begin{aligned} \mathcal{F}_{\psi_0, \tau}(\phi_t) + \mathcal{F}_{\psi_1, \tau}(\phi_{1-t}) &\leq \mathcal{F}_{\psi_0, \tau}(\phi_0) + \mathcal{F}_{\psi_1, \tau}(\phi_1) \\ &\quad - \frac{t}{2\tau} (d^2(\phi_0, \phi_1) - d^2(\psi_0, \psi_1) + (d(\phi_0, \phi_1) - d(\psi_0, \psi_1))^2) \\ &\quad + \frac{t^2}{\tau} d^2(\phi_0, \phi_1) \\ &\leq \mathcal{F}_{\psi_0, \tau}(\phi_t) + \mathcal{F}_{\psi_1, \tau}(\phi_{1-t}) \\ &\quad - \frac{t}{2\tau} (d^2(\phi_0, \phi_1) - d^2(\psi_0, \psi_1) + (d(\phi_0, \phi_1) - d(\psi_0, \psi_1))^2) \\ &\quad + \frac{t^2}{\tau} d^2(\phi_0, \phi_1) \end{aligned}$$

where the second inequality follows from the definition of the resolvent operator. It follows that

$$0 \leq -d^2(\phi_0, \phi_1) + d^2(\psi_0, \psi_1) - (d(\phi_0, \phi_1) - d(\psi_0, \psi_1))^2 + 2td^2(\phi_0, \phi_1).$$

Rearranging and sending  $t$  to zero gives the result.  $\square$

Lastly, one can characterize some convergence properties of minimizing movements, which bear some relationship to Conjecture 1.2.

**Theorem 5.23.** ([24] Proposition 2.40) *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold.*

- (1) *If there exists a cscK metric in  $[\omega]$ , then  $d(\phi_t, \phi_0)$  remains bounded for every  $\phi_0 \in \overline{\mathcal{H}}$ ,  $t > 0$ .*

- (2) If there exists  $\phi_0 \in \overline{\mathcal{H}}$  and a sequence  $t_n \rightarrow \infty$  so that  $d(\phi_{t_n}, \phi_0) \leq C$ , then there exists a minimizer for  $\overline{\nu}$ .
- (3) If there exists  $\phi_0 \in \overline{\mathcal{H}}$  and a sequence  $t_n \rightarrow \infty$  so that  $\{\phi_{t_n}\} \rightarrow \phi_\infty$ , then  $\phi_\infty$  is a minimizer for  $\overline{\nu}$ .

**Remark 5.24.** It is possible ([24] Theorem 2.42) to guarantee a priori convergence to a fixed point if one assumes *uniform convexity* for  $\overline{\nu}$ . That is, there exists  $\epsilon > 0$  so that for  $\phi : [0, 1] \rightarrow \overline{\mathcal{H}}$  a geodesic,

$$\overline{\nu}(\phi_t) \leq (1-t)\overline{\nu}(\phi_0) + t\overline{\nu}(\phi_1) - \epsilon t(1-t)\overline{d}(\phi_0, \phi_1).$$

By examining the second variation formula for  $K$ -energy, one sees that such an inequality could potentially be shown given some a priori control over the lowest eigenvalue of the Lichnerowicz Laplacian. As this operator has a kernel if and only if the manifold admits holomorphic vector fields, one sees here how the presence or lack of such vector fields influences the convergence of Calabi flow.

## 6. HIGHER REGULARITY OF MINIMIZING MOVEMENTS

In this section we prove Theorem 1.5. First we derive a priori estimates for Kähler potentials in geodesic balls in the intersection of geodesic balls of  $\mathcal{H}$  and sublevel sets of  $\nu$ . We use these to control large time steps of discrete solutions to Calabi flow, and then pass these estimates to the limiting minimizing movement.

**Lemma 6.1.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. There exists a constant  $C$  such that for all  $\phi \in \mathcal{H}$ ,*

$$J(\phi) \geq -C\sqrt{V}d(0, \phi).$$

*Proof.* Fix  $\psi \in \mathcal{H}$ , and let  $\gamma : [0, 1] \rightarrow \mathcal{H}$  be the unique  $C^{1,1}$  geodesic connecting 0 to  $\psi$ . We note that, if  $C$  denotes a lower bound for the Ricci curvature of  $\omega$ ,

$$\begin{aligned} \left. \frac{d}{dt} J \right|_{t=0} &= - \int_M \left. \frac{\partial \gamma}{\partial t} \right|_{t=0} \rho(\omega) \wedge \omega^{n-1} \\ &\geq -C \int_M \left| \left. \frac{\partial \gamma}{\partial t} \right|_{t=0} \right| \omega^n \\ &\geq -C \left\| \left. \frac{\partial \gamma}{\partial t} \right|_{t=0} \right\|_{L^2(\omega)} \sqrt{V} \\ &= -CE(0)^{\frac{1}{2}} \sqrt{V} \\ &= -Cd(0, \phi) \sqrt{V}, \end{aligned}$$

where the last equality follows since the energy element along a geodesic is constant. By ([5] Proposition 2),  $J$  is convex along  $C^{1,1}$  geodesics, and hence

$$\frac{d}{dt} J \geq -C\sqrt{V}d(0, \phi)$$

for all  $t \in [0, 1]$ . Integrating this inequality over  $[0, 1]$  yields the proposition.  $\square$

**Lemma 6.2.** *There exists a constant  $C$  so that for all  $\phi \in \mathcal{H}$ ,*

$$(6.1) \quad \sup_M \phi \leq \frac{1}{V} I^A(\phi) + \frac{1}{\sqrt{V}} d(0, \phi) + C.$$

*Proof.* Since  $\text{tr}_\omega (\omega + \sqrt{-1} \partial \bar{\partial} \phi) = n + \Delta_g \phi > 0$ , we can integrate against the Greens function for  $g$  to yield

$$\begin{aligned} \phi(x) &= \frac{1}{V} \int_M \phi(y) \omega^n(y) - \frac{1}{V} \int_M \Delta \phi G(x, y) \omega^n(y) \\ &\leq \frac{1}{V} \int_M \phi \omega^n + \frac{n}{V} \int_M G(x, y) \omega^n(y) \\ &\leq \frac{1}{V} \int_M \phi \omega^n + C \\ &= \frac{1}{V} \left( I^A(\phi) + \int_M \phi \omega_\phi^n \right) + C \\ &\leq \frac{1}{V} \left( I^A(\phi) + \int_{\phi > 0} \phi \omega_\phi^n \right) + C \\ &\leq \frac{1}{V} I^A(\phi) + \frac{1}{\sqrt{V}} d(0, \phi) + C, \end{aligned}$$

where the last line follows from Lemma 3.7.  $\square$

**Lemma 6.3.** *Given  $\phi \in \mathcal{H}$  such that  $I(\phi) = 0$  one has*

$$\|\phi\|_{L^2(\omega)}^2 \leq C(1 + d(0, \phi)) + I^A(\phi).$$

*Proof.* Let  $\gamma : [0, 1] \rightarrow \mathcal{H}$  denote the unique  $C^{1,1}$  geodesic connecting 0 to  $\phi$ . Since  $\ddot{\gamma} \geq 0$ , arguing as in (3.3) we have that

$$\phi \geq \dot{\gamma}(0).$$

It follows that if  $\phi_- = -\inf\{0, \phi\}$ , we have  $\phi_-^2 \leq (\dot{\gamma}(0))^2$ . Since  $\sup \phi \leq C + \frac{d(0, \phi)}{\sqrt{V}} + \frac{1}{V} I^A(\phi)$  by Lemma 6.2, we have that

$$\begin{aligned} \|\phi\|_{L^2(\omega)}^2 &= \int_M (\phi_+^2 + \phi_-^2) \omega^n \\ &\leq \int_M \left( \left( C + \frac{d(0, \phi)}{\sqrt{V}} + \frac{1}{V} I^A(\phi) \right) + |\dot{\gamma}(0)|^2 \right) \omega^n \\ &\leq CV + \sqrt{V} d(0, \phi) + E(\gamma(0)) + I^A(\phi) \\ &\leq C + Cd(0, \phi) + I^A(\phi). \end{aligned}$$

$\square$

Next we recall two lemmas from the work of Tian.

**Lemma 6.4.** ([28] Proposition 2.1) *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. There exist  $\alpha, C > 0$  such that for all  $\phi \in \mathcal{H}$  satisfying  $\sup_M \phi = 0$  one has*

$$\int_M e^{-\alpha\phi} \omega^n \leq C.$$

**Lemma 6.5.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. There exist  $\alpha, C > 0$  such that for all  $\phi \in \mathcal{H}$  one has*

$$\frac{1}{V} \int_M \log \left( \frac{\omega_\phi^n}{\omega^n} \right) \omega_\phi^n \geq \frac{\alpha}{V} I^A(\phi) - C.$$

*Proof.* The proof is contained in ([29] pg. 95). From Lemma 6.4 there exists  $C, \alpha$  such that

$$\frac{1}{V} \int_M \exp \left[ -\log \frac{\omega_\phi^n}{\omega^n} - \alpha(\phi - \sup \phi) \right] \omega_\phi^n \leq C.$$

Since the exponential function is convex we can apply Jensen's inequality to yield

$$\begin{aligned} \frac{1}{V} \int_M \log \left( \frac{\omega_\phi^n}{\omega^n} \right) \omega_\phi^n &\geq -\frac{\alpha}{V} \int_M (\phi - \sup \phi) \omega_\phi^n - \log C \\ &\geq -\frac{\alpha}{V} \left( \int_M \phi \omega_\phi^n + V \sup \phi \right) - C \\ &= \frac{\alpha}{V} \left( I^A(\phi) - \int_M \phi \omega^n + V \sup \phi \right) - C \\ &\geq \frac{\alpha}{V} \left( I^A(\phi) - \sup \phi \int_M \omega^n + V \sup \phi \right) - C \\ &= \frac{\alpha}{V} I^A(\phi) - C. \end{aligned}$$

□

**Proposition 6.6.** *Let  $(M^{2n}, \omega, J)$  be a compact Kähler manifold. Given  $A, B > 0$  there exists  $C(A, B) > 0$  such that if  $\phi \in \mathcal{H}$  satisfies*

- $I(\phi) = 0$
- $-A \leq \nu(\phi) \leq A$
- $d(0, \phi) \leq B$ ,

*then*

- (1)  $-C \leq J(\phi) \leq C$ ,
- (2)  $I^A(\phi) \leq C$ ,
- (3)  $J^A(\phi) \leq C$ ,
- (4)  $-C \leq \int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n \leq C$ .

*Proof.* By Lemma 6.1, we obtain the lower bound

$$J(\phi) \geq -CB^2.$$

On the other hand, using the representation of  $K$ -energy in (2.10) and Lemma 6.5 we have

$$\begin{aligned} J(\phi) &= \nu(\phi) - \int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n \\ &\leq A - \alpha I^A(\phi) + C \\ &\leq A + C. \end{aligned}$$

Since  $J(\phi)$  is thus bounded above and below, turning again to (2.10) yields an upper and lower bound on  $\int_M \log \frac{\omega_\phi^n}{\omega^n} \omega_\phi^n$ . Applying Lemma 6.5 again yields the upper bound for  $I^A(\phi)$ .  $\square$

*Proof of Theorem 1.5.* Fix  $\phi_0 \in \mathcal{H}$  and let  $\phi : [0, \infty) \rightarrow \overline{\mathcal{H}}$  be the  $K$ -energy minimizing movement with initial condition  $\phi_0$ . As guaranteed by (5.3), we know that

$$\phi_t = \lim_{n \rightarrow \infty} W_{\frac{t}{n}}^n(\phi_0).$$

It follows from Lemma 5.8 that  $d(\phi_0, \phi_t)$  is uniformly controlled in terms of  $\nu(\phi_0)$  and  $t$ . Now choose a sequence  $\{\phi_t^j\} \in \mathcal{H}$  converging to  $\phi_t$  in the distance topology. Certainly  $d(\phi_0, \phi_t^j)$  is then uniformly controlled in terms of  $\nu(\phi_0)$  and  $t$ , and thus  $d(0, \phi_t^j)$  is controlled in terms of  $d(0, \phi_0), \nu(\phi_0)$  and  $t$  by the triangle inequality. Combining this with Lemma 6.2, Lemma 6.3 and Proposition 6.6 there exists a constant  $C = C(t, d(0, \phi_0), \nu(\phi_0))$  such that, for all  $j$ ,

$$\|\phi_t^j\|_{H_1^2} + \sup_M \phi_t^j \leq C.$$

By choosing a subsequence we obtain a sequence satisfying these inequalities and converging weakly in  $H_1^2$  and almost everywhere, and so both of these inequalities pass to the limit as  $j \rightarrow \infty$ , finishing the theorem.  $\square$

## 7. SMOOTH CONVERGENCE OF DISCRETE CALABI FLOWS WITH UNIFORM BOUNDS

In this section we prove Theorem 1.6, which says that a sequence of discrete Calabi flows with given initial condition, vanishing step size, and uniform  $C^{4,\alpha}$  bounds in space contains a subsequence which converges to a smooth solution of Calabi flow.

**Theorem 7.1.** *Suppose that given  $T > 0$  there exists  $C, \alpha > 0$  and a sequence  $\{\phi_j^i\}$  of discrete Calabi flows on  $[0, T]$  with uniform step size  $\tau_i \rightarrow 0$  and initial condition  $\phi_0$ , such that*

$$(7.1) \quad \sup_{i,j} |\phi_j^i|_{C^{4,\alpha}} + \left| \log \frac{\omega_{\phi_j^i}^n}{\omega^n} \right|_{C^{2,\alpha}} \leq C,$$

$$(7.2) \quad \sup_{i,j} \left| \phi_{j+1}^i - \phi_j^i - \frac{\partial \gamma_j^i}{\partial t} \right|_{C^0} \leq \tau_i o(\tau_i).$$

*Then there exists a subsequence of  $\{\phi_t^\epsilon\}$  converging in  $C^{4,\alpha'}$ ,  $\alpha' < \alpha$ , to a smooth solution of Calabi flow on  $[0, T]$ .*

**Remark 7.2.** The hypothesis (7.2) is reasonable to make, as it says that when the geodesic distance between two very close point is rescaled to unit length, the curve is approaching a the straight line path between the points, which should follow from uniqueness of solutions to the geodesic equation.

*Proof.* It follows directly from Arzela-Ascoli that at any time  $t \in [0, T]$  one obtains a subsequence converging in  $C^{4,\alpha}$ . This yields a one-parameter family  $\phi_t \in \mathcal{H}$ . We claim that this family is differentiable in  $t$ , and moreover satisfies the Calabi flow. Fix some time  $t_0 \in [0, T]$ , and fix  $h > 0$ . We claim there exists constants  $C, \gamma > 0$  such that

$$(7.3) \quad P(t_0, h) := \lim_{i \rightarrow \infty} \left| \frac{\phi^i(t_0 + h) - \phi^i(t_0)}{h} - (s_{\phi^i(t_0)} - \bar{s}) \right| = o(h).$$

Since we have convergence of  $\phi_t^i$  to  $\phi_t$  in  $C^{4,\alpha}$ , the theorem will follow from this claim. As each  $\phi^i$  is a discrete Calabi flow with uniform step size, we have that  $\phi^i(t) = \phi_{\lfloor \frac{t}{\tau_i} \rfloor}^i$ . Recall also the variational equation

$$(7.4) \quad \frac{1}{\tau_i} \frac{\partial \gamma_j^i}{\partial t} \Big|_{t=1} = s_{\phi_{j+1}^i} - \bar{s}.$$

where  $\gamma_j^i : [0, 1] \rightarrow \mathcal{H}$  is the unique  $C^{1,1}$  geodesic connecting  $\phi_j^i$  and  $\phi_{j+1}^i$ . Hence we obtain

$$\begin{aligned} P(t_0, h) &= \lim_{i \rightarrow \infty} \left| \frac{\phi_{\lfloor \frac{t_0+h}{\tau_i} \rfloor}^i - \phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i}{h} - \left( s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} - \bar{s} \right) \right| \\ &= \lim_{i \rightarrow \infty} \frac{1}{h} \left| \left[ \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \phi_{j+1}^i - \phi_j^i \right] - h \left( s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} - \bar{s} \right) \right| \\ &= \lim_{i \rightarrow \infty} \frac{1}{h} \left| \left[ \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \frac{\partial \gamma_j^i}{\partial t} \right] - h \left( s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} - \bar{s} \right) + E \right| \\ &\leq \lim_{i \rightarrow \infty} \frac{1}{h} \left| \left[ \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \frac{\partial \gamma_j^i}{\partial t} \right] - h \left( s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} - \bar{s} \right) \right| + \lim_{i \rightarrow \infty} \frac{1}{h} |E| \\ &=: A + B. \end{aligned}$$

where  $E$  is defined by the equality in the line in which it appears, and  $A$  and  $B$  are the two terms appearing in the penultimate line. First we estimate term  $B$  using (7.2).



$$\begin{aligned}
(7.5) \quad B &= \lim_{i \rightarrow \infty} \frac{1}{h} \left\| \left[ \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \phi_{j+1}^i - \phi_j^i - \frac{\partial \gamma_j^i}{\partial t} \right] \right\| \\
&\leq \lim_{i \rightarrow \infty} \frac{1}{h} \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \left| \phi_{j+1}^i - \phi_j^i - \frac{\partial \gamma_j^i}{\partial t} \right| \\
&\leq \lim_{i \rightarrow \infty} \frac{C}{h} \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \tau_i o(\tau_i) \\
&= \lim_{i \rightarrow \infty} \frac{C}{h} \left( \frac{h}{\tau_i} \right) \tau_i o(\tau_i) \\
&= 0.
\end{aligned}$$

Turning to  $A$  we first prove a lemma.

**Lemma 7.3.** *Given  $0 < \alpha' < \alpha < 1, C > 0$  and  $\epsilon > 0$  there exists  $\delta > 0$  so that if  $\phi_1, \phi_2 \in \mathcal{H}$  satisfy*

$$(1) \quad |\phi_i|_{C^{4,\alpha}} + \left| \log \frac{\omega_{\phi_i}^n}{\omega^n} \right|_{C^{2,\alpha}} \leq C,$$

$$(2) \quad d(\phi_1, \phi_2) \leq \delta,$$

then  $|\phi_1 - \phi_2|_{C^{4,\alpha}} \leq \epsilon$ .

*Proof.* If the statement were false then we can choose a sequence  $\{\delta_i\}, \delta_i \rightarrow 0$ , and sequences of functions  $\{\phi_1^i, \phi_2^i\}$  satisfying the hypotheses but  $|\phi_1^i - \phi_2^i|_{C^{4,\alpha}} > \epsilon$ . By property (1) we may apply Arzela-Ascoli to obtain a subsequence of  $\{\phi_1^i\}$  converging in  $C^{4,\alpha'}$ ,  $\alpha' < \alpha$  to  $\phi_1^\infty$ , and likewise one has  $\phi_2^\infty$ . By property (2) and the estimates of the Calabi-Yau theorem [30], one obtains a uniform lower bound on  $\omega_{\phi_j^i}$ , and so  $\phi_1^\infty, \phi_2^\infty \in \mathcal{H}$ . Since  $d(\phi_1^i, \phi_2^i) \leq \delta_i \rightarrow 0$ , given the uniform bounds on the metrics  $\{\phi_j^i\}$  we conclude from Lemma 3.7 that

$$\lim_{i \rightarrow \infty} \|\phi_1^i - \phi_2^i\|_{L^1(\omega)} = 0.$$

It follows that  $\phi_1^\infty = \phi_2^\infty$  and so for sufficiently large  $i$  one has  $\|\phi_1^i - \phi_2^i\|_{C^{4,\alpha'}} \leq \epsilon$ , a contradiction.  $\square$

Now note that since the summand defining  $A$  consists of  $\frac{h}{\tau_i}$  terms, we can re-express

$$A = \lim_{i \rightarrow \infty} \frac{1}{h} \left\| \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \left[ \frac{\partial \gamma_j^i}{\partial t} - \tau_i \left( s_{\phi_j^i}^{\lfloor \frac{t_0}{\tau_i} \rfloor} - \bar{s} \right) \right] \right\|$$

Then, inserting the variational equation (7.4) and applying the triangle inequality we have

$$A = \lim_{i \rightarrow \infty} \frac{1}{h} \left| \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \tau_i \left( s_{\phi_{j+1}^i} - s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} \right) \right| \leq \lim_{i \rightarrow \infty} \frac{\tau_i}{h} \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \left| s_{\phi_{j+1}^i} - s_{\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i} \right|.$$

Next we observe that, by Lemma 5.8, for all  $j$  one has  $\phi_{j+1}^i \in B_{C\tau}(\omega_{\phi_j^i})$ . Thus for every  $j \in \left[ \lfloor \frac{t_0}{\tau_i} \rfloor, \lfloor \frac{t_0+h}{\tau_i} \rfloor - 1 \right]$  there is a piecewise geodesic curve connecting  $\phi_j^i$  to  $\phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i$  consisting of at most  $\frac{h}{\tau_i}$  segments each of length no greater than  $\tau_i$ . It follows by the triangle inequality that for all such  $j$ ,

$$d(\phi_j^i, \phi_{\lfloor \frac{t_0}{\tau_i} \rfloor}^i) \leq Ch.$$

Again using that the summand describing  $A$  consists of  $\frac{h}{\tau_i}$  terms, it follows from Lemma 7.3 that

$$A \leq \lim_{i \rightarrow \infty} \frac{\tau_i}{h} \sum_{j=\lfloor \frac{t_0}{\tau_i} \rfloor}^{\lfloor \frac{t_0+h}{\tau_i} \rfloor - 1} \epsilon(Ch) = \epsilon(Ch).$$

This completes the proof of (7.3), finishing the theorem.  $\square$

## 8. CONCLUSION

As is clear from the proof of Theorem 1.5, more is proved in the sense that minimizing sequences for each Moreau-Yosida functional satisfy all of the estimates of Proposition 6.6. Obtaining further regularity results in this direction is an essential step in overcoming the gap between Theorem 1.3 and Conjecture 1.1. Many ingenious arguments are exploited in Chen-Tian's proof of uniqueness of cscK metrics, which ultimately is a regularity proof, showing that the geodesic connecting two critical points of  $\nu$  is itself smooth. The metrics in play in this proof already satisfy an a priori  $C^{1,1}$  bound though, making the problem more tractable. Ultimately, obtaining stronger a priori estimates on the intersection of geodesic balls and sublevel sets of  $\nu$  will be essential in obtaining higher regularity of minimizing movements.

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